

STABILITY OF INTEGRABLE AND NONINTEGRABLE STRUCTURES

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ABSTRACT. In this paper we give a comprehensive account of several recent results on the stability of nontrivial soliton structures for some well-known non periodic dispersive models. We will focus on the simpler case of the generalized Korteweg-de Vries equations, covering the classical stability results by Bona, Souganidis, Strauss until the results by Martel and Merle and our recent collaborations with Miguel Alejo and Luis Vega.

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1. INTRODUCTION. CLASSICAL STABILITY OF SOLITONS

1.1. Introduction. Consider the generalized Korteweg-de Vries (gKdV) equation on the real line

$$u_t + (u_{xx} + u^p)_x = 0, \quad (1.1)$$

(p is a positive integer) where $u = u(t, x)$ is a real-valued function, and $(t, x) \in \mathbb{R}^2$. The case $p = 2$ is the famous Korteweg-de Vries (KdV) equation, a model for shallow water waves and a canonical example of a dispersive nonlinear evolution equation. We assume p integer since u may have no definite sign, otherwise we must consider the odd power case $|u|^{p-1}u$, $p \geq 1$.

Even if (1.1) is a simple 1d model, its study proved to be a very difficult problem (see e.g. the monograph by Linares and Ponce [33]). In particular, if one considers the Cauchy problem

$$u_t + (u_{xx} + u^p)_x = 0, \quad u(t = 0) = u_0, \quad (1.2)$$

Kato first, and then Kenig, Ponce and Vega [27] showed global existence and well-posedness if $u_0 \in H^1(\mathbb{R})$, for $p = 2, 3$ and 4. These results have been improved by many others, see e.g. the works by Bourgain [13] the I-team [16]. For the purposes of this course, we only need the H^1 global well-posedness, that is

Theorem 1.1 ([27]). *Assume $u_0 \in H^1(\mathbb{R})$, with $p = 2, 3$ or 4 in (1.2). Then there exists a unique $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ solution of (1.2) in the Duhamel sense:*

$$u(t) = S(t)u_0 - \int_0^t S(t-s)[(u^p)_x(s)]ds, \quad S(t) := e^{-t\partial_x^3}.$$

Moreover, one has

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1(\mathbb{R})} \leq C(\|u_0\|_{H^1(\mathbb{R})}).$$

Whenever $p \geq 5$, problem (1.2) becomes a very difficult one: in particular, if $p = 5$ it is well-known that there are blowing-up solutions (see Martel and Merle [37]). If $p > 5$, very little is known, mainly because the problem is L^2 “supercritical”. We will discuss some of these affirmations below.

1.2. Solitons. In addition to the previous global well-posedness result, (1.2) is interesting by its *soliton* solutions. A soliton for (1.2) is a solution of the form

$$u(t, x) = Q_c(x - ct - x_0), \quad c > 0, x_0 \in \mathbb{R}. \quad (1.3)$$

Note that Q_c is a fixed profile depending only on $c > 0$ (usually referred as the scaling, or the velocity of the soliton), and x_0 is a free shift parameter. Replacing (1.3) in (1.1), and assuming that Q_c vanishes at infinity, we find that Q_c must satisfy the equation

$$(Q_c'' - cQ_c + Q_c^p)' = 0,$$

or

$$Q_c'' - cQ_c + Q_c^p = 0 \quad \text{in } \mathbb{R}. \quad (1.4)$$

The solutions to this elliptic equation (actually, it is an ODE), are well-understood: $Q_c = Q_c(s)$ is given by the explicit formula

$$Q_c(s) = c^{1/(p-1)} Q(\sqrt{c}s),$$

where¹

$$Q(s) (= Q_{c=1}) := \left(\frac{p+1}{2 \cosh^2(\frac{p-1}{2}s)} \right)^{1/(p-1)}.$$

In other words, a soliton is a traveling wave solution with positive speed $c > 0$. Note that the bigger $c > 0$ is, the faster the soliton travels.

We would like to emphasize that solitons are purely nonlinear objects: the associated linear equation, denoted as the Airy equation

$$u_t + u_{xxx} = 0,$$

has no soliton solutions.

A very nice relation between solitons and the global well-posedness result can be established using the L^2 norm of each soliton. We have

$$\|Q_c\|_{L^2(\mathbb{R})} \sim c^{\frac{1}{p-1} - \frac{1}{4}}, \quad (1.5)$$

so that $\frac{1}{p-1} - \frac{1}{4} > 0$ if $p = 2, 3, 4$ (L^2 -subcritical regime), $\frac{1}{p-1} - \frac{1}{4} = 0$ if $p = 5$ (L^2 -critical regime), and $\frac{1}{p-1} - \frac{1}{4} < 0$ if $p > 5$ (L^2 -supercritical regime).

In other words, in a supercritical problem, small solitons (in the sense of the L^∞ norm), are very large in the L^2 norm. Whenever $p = 5$, all solitons have the same size. We can summarize these properties by saying that

$$\partial_c \int_{\mathbb{R}} Q_c^2 \Big|_{c=1} =: \int_{\mathbb{R}} \Lambda Q Q \begin{cases} > 0, & p = 2, 3, 4, \\ = 0, & p = 5, \\ < 0, & p > 5. \end{cases}$$

Here we have denoted by ΛQ_c the so-called scaling direction, namely

$$\Lambda Q_c(s) := \partial_c Q_c(s) = \frac{1}{c} \left(\frac{1}{p-1} Q_c(s) + \frac{1}{2} s Q'_c(s) \right), \quad (1.6)$$

and $\Lambda Q := \Lambda Q_c \Big|_{c=1}$. Note that ΛQ_c is an even function² which satisfies the equation (just take derivative with respect to c in (1.4))

$$\mathcal{L} \Lambda Q_c = -Q_c. \quad (1.7)$$

1.3. Stability of solitons. In this paragraph we will discuss the stability problem for soliton solutions. Assume that $u_0 \in H^s(\mathbb{R})$ satisfies

$$\|u_0 - Q_c\|_{H^s} < \alpha, \quad (1.8)$$

where $\alpha \ll 1$ and $s \geq 0$.

Definition 1.2. We say that Q_c is (nonlinearly) stable in H^s if under (1.8) one has

$$\sup_{t \in \mathbb{R}} \|u(t) - Q_c(\cdot - \rho(t))\|_{H^s} \lesssim \alpha, \quad (1.9)$$

for some $\rho(t) \in \mathbb{R}$. Otherwise we say that Q_c is unstable.

¹Note that Q_c is even.

²In fact, ΛQ_c is a very important direction for the dynamics of nontrivial perturbations of a soliton solution, for instance in the critical case $p = 5$.

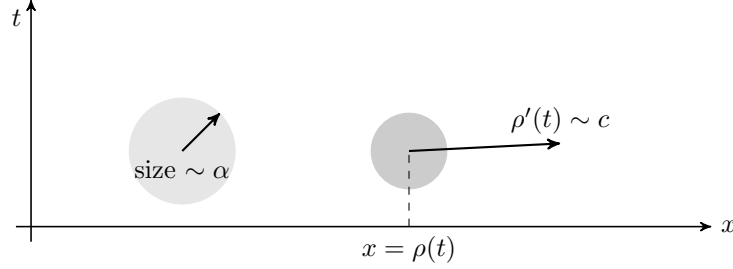


FIGURE 1. Faster solitons are darker and more concentrated; speed is commensurate with arrow length. Here a small soliton perturbs the big one, which remains stable up to a modification of the center of mass.

Note that the constant involved in estimate (1.9) does not depend on t and α . The parameter $\rho(t)$ is absolutely necessary since if $c \sim c'$, with $|c - c'| = \alpha \ll 1$, one has

$$\|Q_c - Q_{c'}\|_{H^s} \sim \alpha \ll 1,$$

but the corresponding solutions satisfy

$$\|Q_c(\cdot - ct) - Q_{c'}(\cdot - c't)\|_{H^s} \sim 1,$$

as $t \rightarrow \infty$. In that sense, we say that (1.9) is a sort of *orbital* stability result (see Fig. 1).

Theorem 1.3 (H^1 stability, [12]). *Assume $p = 2, 3, 4$, $c > 0$, $x_0 \in \mathbb{R}$. There are $\alpha_0 > 0$, $C_0 > 0$ such that for all $\alpha \in (0, \alpha_0)$ and for all $u_0 \in H^1(\mathbb{R})$ such that*

$$\|u_0 - Q_c(\cdot - x_0)\|_{H^1} < \alpha,$$

one has

$$\sup_{t \in \mathbb{R}} \|u(t) - Q_c(\cdot - \rho(t))\|_{H^s} \leq C_0 \alpha,$$

for some $\rho(t)$ which satisfies the estimate

$$\sup_{t \in \mathbb{R}} |\rho'(t) - c| \leq CC_0 \alpha,$$

for some $C > 0$.

The previous theorem was proved by Benjamin [10] and Bona, Souganidis and Strauss [12]. The proof that we present here is based in the approach introduced by M. Weinstein [59].

1.4. Proof of Theorem 1.3. The key idea is the use of conservation laws. Recall that (1.1) has at least two conserved quantities: the mass

$$M[u](t) := \frac{1}{2} \int_{\mathbb{R}} u^2(t, x) dx = M[u_0], \quad (1.10)$$

and the energy

$$E[u](t) := \frac{1}{2} \int_{\mathbb{R}} u_x^2(t, x) dx - \frac{1}{p+1} \int_{\mathbb{R}} u^{p+1}(t, x) dx = E[u_0]. \quad (1.11)$$

Both quantities are preserved by the H^1 flow. Now we analyze the evolution of a solution $u(t)$ which is a small perturbation of a soliton: if for some $\rho(t)$ fixed we have

$$u(t, x) = Q_c(x - \rho(t)) + z(t, x),$$

(note that $z(t, x)$ depends on the definition of $\rho(t)$) then

$$\begin{aligned} E[u](t) &= E[Q_c] + \int_{\mathbb{R}} Q'_c z_x - \int_{\mathbb{R}} Q_c^p z \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} z_x^2 - \frac{p}{2} \int_{\mathbb{R}} Q_c^{p-1} z^2 + O(\|z(t)\|_{H^1(\mathbb{R})}^3). \end{aligned}$$

Note that for the first order term on z we have, using (1.3),

$$\int_{\mathbb{R}} Q'_c z_x - \int_{\mathbb{R}} Q_c^p z = -c \int_{\mathbb{R}} Q_c z,$$

which can be cancelled if we use the mass

$$cM[u](t) = cM[Q_c] + c \int_{\mathbb{R}} Q_c z + \frac{c}{2} \int_{\mathbb{R}} z^2.$$

Therefore, we have

$$\begin{aligned} E[u_0] + cM[u_0] &= E[Q_c] + cM[Q_c] \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} z_x^2 + \frac{c}{2} \int_{\mathbb{R}} z^2 - \frac{p}{2} \int_{\mathbb{R}} Q_c^{p-1} z^2 + O(\|z(t)\|_{H^1(\mathbb{R})}^3). \end{aligned} \quad (1.12)$$

Now we consider the quadratic term on z above. It is not difficult to see that

$$\frac{1}{2} \int_{\mathbb{R}} z_x^2 + \frac{c}{2} \int_{\mathbb{R}} z^2 - \frac{p}{2} \int_{\mathbb{R}} Q_c^{p-1} z^2 = \int_{\mathbb{R}} z \mathcal{L} z,$$

where

$$\mathcal{L} z := -z_{xx} + cz - pQ_c^{p-1} z. \quad (1.13)$$

Now, using classical analysis of operators (see e.g Reed-Simon, vol. 4), we have

Lemma 1.4. *The following are satisfied.*

- (1) \mathcal{L} is a self-adjoint operator defined on $L^2(\mathbb{R})$, with domain $D(\mathcal{L}) = H^2(\mathbb{R})$.
- (2) $\ker \mathcal{L}$ is spanned by Q'_c (direct from (1.3) and ODE analysis).
- (3) \mathcal{L} has a unique negative eigenvalue $-\lambda_0 < 0$ of multiplicity one. The associated (explicit) eigenfunction χ_c satisfies $\chi_c \in S(\mathbb{R})$ (the Schwartz class), and

$$\mathcal{L}\chi_c = -\lambda_0\chi_c, \quad \|\chi_c\|_{L^2(\mathbb{R})} = 1. \quad (1.14)$$

- (4) The continuum spectrum of \mathcal{L} is the closed interval $[c, +\infty)$.
- (5) Coercivity. There is $\gamma_0 > 0$ such that the following holds. Assume that

$$\int_{\mathbb{R}} \tilde{z} Q'_c = \int_{\mathbb{R}} \tilde{z} \chi_c = 0. \quad (1.15)$$

Then

$$\int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} \geq \gamma_0 \|\tilde{z}\|_{H^1(\mathbb{R})}^2. \quad (1.16)$$

As an immediate corollary of (1.16), one has the following.

Corollary 1.5. *There is $\gamma_0 > 0$ such that the following holds. Assume that*

$$\int_{\mathbb{R}} \tilde{z} Q'_c = 0.$$

Then

$$\int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} \geq \gamma_0 \|\tilde{z}\|_{H^1(\mathbb{R})}^2 - \frac{1}{\gamma_0} \left| \int_{\mathbb{R}} \chi_c \tilde{z} \right|^2. \quad (1.17)$$

For the proof of this result, just modify \tilde{z} with a suitable linear combination of χ_c such that now (1.15) is satisfied.

However, from the original ideas of Weinstein, we will prove the following very useful variation of (1.17).

Lemma 1.6. *There is $\tilde{\gamma}_0 > 0$ such that the following holds. Assume now that*

$$\int_{\mathbb{R}} z Q'_c = 0.$$

Then

$$\int_{\mathbb{R}} z \mathcal{L} z \geq \tilde{\gamma}_0 \|z\|_{H^1(\mathbb{R})}^2 - \frac{1}{\tilde{\gamma}_0} \left| \int_{\mathbb{R}} Q_c z \right|^2. \quad (1.18)$$

Let us assume for a moment the validity of this lemma. Coming back to the proof of the stability result, we are going to use the Implicit Function Theorem in order to ensure that we can fix $\rho(t)$ such that, for all time t ,

$$\int_{\mathbb{R}} z(t, x) Q'_c(x - \rho(t)) dx = 0, \quad (1.19)$$

so that we fix now $z(t, x)$. Usually we call this method *modulation*. This can be done provided $z(t)$ is small, which is true at least for some time $0 < t \leq T_0$. The idea is to prove that we can take $T_0 = +\infty$ by a simple bootstrap argument. Note that (1.19) is justified since the smooth functional

$$H^1(\mathbb{R}) \times \mathbb{R} \ni (u, \rho) \mapsto \int_{\mathbb{R}} (u(x) - Q_c(x - \rho)) Q'_c(x - \rho) dx \in \mathbb{R}$$

has nondegenerate partial derivative with respect to ρ at the point (Q_c, ρ) , because

$$\int_{\mathbb{R}} Q_c'^2 \neq 0.$$

Therefore, in a small H^1 -neighborhood of Q_c one has (1.19) well-defined (the uniformity in time for $u(t) \in H^1(\mathbb{R})$ can be assured at least for a suitable amount of time that will be bootstrapped later on). Using (1.18) applied to $z(t, x)$ we have

$$\int_{\mathbb{R}} z(t) \mathcal{L} z(t) \geq \gamma_0 \|z(t)\|_{H^1(\mathbb{R})}^2 - \frac{1}{\gamma_0} \left| \int_{\mathbb{R}} Q_c z(t) \right|^2.$$

Now we evaluate (1.12) at time $t = 0$ and some $t \leq T_0$. We have (we emphasize that the constants involved are independent of time)

$$\|z(t)\|_{H^1(\mathbb{R})}^2 \lesssim \left| \int_{\mathbb{R}} Q_c z(t) \right|^2 + \|z(0)\|_{H^1(\mathbb{R})}^2 + \|z(t)\|_{H^1(\mathbb{R})}^3.$$

Note that if α_0 is chosen smaller, the last term on the right above can be absorbed by the left hand side. We are left to give a suitable estimate on the first term on the right. But now we use the conservation of mass: we have

$$\int_{\mathbb{R}} z Q_c(t) = \int_{\mathbb{R}} z Q_c(0) + \frac{1}{2} \int_{\mathbb{R}} z^2(0) - \frac{1}{2} \int_{\mathbb{R}} z^2(t),$$

so that

$$\left| \int_{\mathbb{R}} z Q_c(t) \right| \lesssim \|z(0)\|_{H^1(\mathbb{R})} + \|z(t)\|_{H^1(\mathbb{R})}^2.$$

Note that this estimate is better than the usual Cauchy-Schwarz estimate for the term in the left-hand side, since the term of order one in $z(t)$ has disappeared. We finally get

$$\|z(t)\|_{H^1(\mathbb{R})} \lesssim \|z(0)\|_{H^1(\mathbb{R})},$$

so that the estimate on $z(t)$ does not depend on t , improving the previous assumption $T_0 < +\infty$. This proves the result.

Now we prove Lemma 1.6.

Proof. We assume that

$$\int_{\mathbb{R}} \tilde{z} Q'_c = \int_{\mathbb{R}} z Q_c = 0.$$

We want to show that

$$\int_{\mathbb{R}} z \mathcal{L} z \geq \tilde{\gamma}_0 \|z\|_{H^1(\mathbb{R})}^2,$$

for some $\tilde{\gamma}_0 > 0$ independent of z . We decompose z and ΛQ_c (cf. (1.6)) as follows

$$z = \beta_0 \chi_c + \tilde{z}, \quad \beta_0 \in \mathbb{R}, \quad \int_{\mathbb{R}} \tilde{z} \chi_c = 0,$$

and

$$\Lambda Q_c = \beta_1 \chi_c + \widetilde{\Lambda Q_c}, \quad \beta_1 \in \mathbb{R}, \quad \int_{\mathbb{R}} \widetilde{\Lambda Q_c} \chi_c = 0.$$

Note that we do not need any component in the Q'_c direction since z and ΛQ_c are orthogonal to this function.

Now, using (1.14) and the previous decomposition, we have

$$\int_{\mathbb{R}} z \mathcal{L} z = -\beta_0^2 \lambda_0 + \int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z}. \quad (1.20)$$

Since $p < 5$,

$$\int_{\mathbb{R}} Q_c \Lambda Q_c > 0, \quad (1.21)$$

and

$$0 > - \int_{\mathbb{R}} Q_c \Lambda Q_c = \int_{\mathbb{R}} \Lambda Q_c \mathcal{L} \Lambda Q_c = -\beta_1^2 \lambda_0 + \int_{\mathbb{R}} \widetilde{\Lambda Q_c} \mathcal{L} \widetilde{\Lambda Q_c}. \quad (1.22)$$

Finally,

$$0 = \int_{\mathbb{R}} z Q_c = - \int_{\mathbb{R}} z \mathcal{L} \Lambda Q_c = \beta_0 \beta_1 \lambda_0 - \int_{\mathbb{R}} \widetilde{\Lambda Q_c} \mathcal{L} \tilde{z},$$

so that

$$\beta_0^2 \beta_1^2 \lambda_0^2 = \left(\int_{\mathbb{R}} \widetilde{\Lambda Q_c} \mathcal{L} \tilde{z} \right)^2. \quad (1.23)$$

From (1.20), (1.22) and (1.23), we have

$$\int_{\mathbb{R}} z \mathcal{L} z = \int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} - \frac{\left(\int_{\mathbb{R}} \Lambda \widetilde{Q_c} \mathcal{L} \tilde{z} \right)^2}{\int_{\mathbb{R}} \Lambda \widetilde{Q_c} \mathcal{L} \Lambda \widetilde{Q_c} + \int_{\mathbb{R}} Q_c \Lambda Q_c}. \quad (1.24)$$

We get the estimate

$$\int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} \geq \gamma_0 \|\tilde{z}\|_{H^1(\mathbb{R})}^2 \geq 0.$$

Moreover, the bilinear operator

$$\int_{\mathbb{R}} v \mathcal{L} w, \quad v, w \in \{\chi_c, Q'_c\}^\perp$$

defines an inner product in $L^2(\mathbb{R})^2$. Therefore, we have a Cauchy-Schwarz inequality:

$$\left| \int_{\mathbb{R}} v \mathcal{L} w \right|^2 \leq \left(\int_{\mathbb{R}} v \mathcal{L} v \right) \left(\int_{\mathbb{R}} w \mathcal{L} w \right),$$

with equality if and only if v is parallel to w . Using this information we have

$$\left(\int_{\mathbb{R}} \Lambda \widetilde{Q_c} \mathcal{L} \tilde{z} \right)^2 \leq \left(\int_{\mathbb{R}} \Lambda \widetilde{Q_c} \mathcal{L} \Lambda \widetilde{Q_c} \right) \left(\int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} \right),$$

and now (1.24) becomes

$$\int_{\mathbb{R}} z \mathcal{L} z \geq \left(\int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} \right) \left[1 - \frac{\int_{\mathbb{R}} \Lambda \widetilde{Q_c} \mathcal{L} \Lambda \widetilde{Q_c}}{\int_{\mathbb{R}} \Lambda \widetilde{Q_c} \mathcal{L} \Lambda \widetilde{Q_c} + \int_{\mathbb{R}} Q_c \Lambda Q_c} \right].$$

Since $\int_{\mathbb{R}} Q_c \Lambda Q_c > 0$, we have that

$$1 - \frac{\int_{\mathbb{R}} \Lambda \widetilde{Q_c} \mathcal{L} \Lambda \widetilde{Q_c}}{\int_{\mathbb{R}} \Lambda \widetilde{Q_c} \mathcal{L} \Lambda \widetilde{Q_c} + \int_{\mathbb{R}} Q_c \Lambda Q_c} > \eta_0 > 0,$$

independent of z . We have then

$$\int_{\mathbb{R}} z \mathcal{L} z \geq \eta_0 \int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} \geq \eta_0 \gamma_0 \|\tilde{z}\|_{H^1(\mathbb{R})}^2 \geq 0.$$

Finally, from (1.20) and the previous inequality we have

$$\begin{aligned} \int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} &= \frac{1}{2} \int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} + \frac{1}{2} \int_{\mathbb{R}} \tilde{z} \mathcal{L} \tilde{z} \\ &\geq \frac{1}{2} \eta_0 \gamma_0 \|\tilde{z}\|_{H^1(\mathbb{R})}^2 + \frac{1}{2} \beta_0^2 \lambda_0 \\ &\geq \frac{1}{2} \eta_0 \gamma_0 \|\tilde{z}\|_{H^1(\mathbb{R})}^2 + \frac{\beta_0^2}{C_0} \|\chi_c\|_{H^1(\mathbb{R})}^2 \\ &\geq \tilde{\gamma}_0 \|z\|_{H^1(\mathbb{R})}^2, \end{aligned}$$

as desired. □

Before finishing this section, some remarks are in order:

Remark 1.1. If $p \geq 5$ it is well-known that solitons are *unstable* objects, see Bona-Souganidis-Strauss [12] for the case $p > 5$, and Martel-Merle [36] for the more difficult case $p = 5$. Compare these results with the critical-supercritical nature of the equation for $p \geq 5$ (1.5).

Remark 1.2. One can prove stability using the variational characterization of the soliton as the ground state solution of the elliptic ODE

$$u'' - cu + u^p = 0, \quad u > 0, \quad u \in H^1(\mathbb{R}).$$

See e.g. Cazenave-Lions [15], Grillakis-Statah-Strauss [24], etc.

Remark 1.3. Proving stability in H^s , $s \neq 1$ is a very difficult problem. If $s \in (0, 1)$, there are polynomial bounds (in time) for the growth of the Sobolev norms (see e.g. [55]), however, note that the fact that $\|Q_c\|_{H^s} \sim 1$ destroys the utility of such results. If $s = 0$ and $p = 2$ (namely, we consider the KdV equation with L^2 initial data), Merle and Vega [49] proved L^2 stability using the Miura transformation (so the proof is deeply non variational). Their proof has been adapted to several other integrable problems (see Mizumachi-Tzvetkov [48], Muñoz [51], Alejo-Muñoz-Vega [7], Buckmaster-Koch [14], among others).

Remark 1.4. One can prove convergence to a soliton at infinity in time, a property called *asymptotic stability*. See the works by Pego and Weinstein [52], Martel-Merle [38], and those related to the nonlinear Schrödinger equation, not considered in these notes.

Remark 1.5. One can also prove stability (and instability) of periodic structures, solutions of (1.1), so (1.4) does not hold in general. See e.g the works by Jaime Angulo [8] and collaborators.

2. STABILITY OF THE SUM OF N -SOLITONS. MULTI-SOLITONS

In the previous section we studied the stability problem for a simple soliton Q_c . The purpose of this lecture is to consider the problem when several solitons interact themselves, in a weak form. Indeed, fix $p = 2, 3$ or 4 , consider the scaling parameters

$$0 < c_1 < c_2 < \cdots < c_N, \tag{2.1}$$

and fixed shift parameters

$$x_1^0, x_2^0, \dots, x_N^0. \tag{2.2}$$

We would like to show that any small perturbation of the sum of N solitons at time $t = 0$,

$$\sum_{j=1}^N Q_{c_j}(x - x_j^0), \tag{2.3}$$

where Q_{c_j} is solution of (1.4), leads to a solution $u(t)$ of (1.1) which stays close to the sum of N -solitons, of the form

$$\sum_{j=1}^N Q_{c_j}(x - \rho_j(t)), \quad \rho_j'(t) \sim c_j \tag{2.4}$$

for all time $t \geq 0$. before continuing, some remarks are in order.

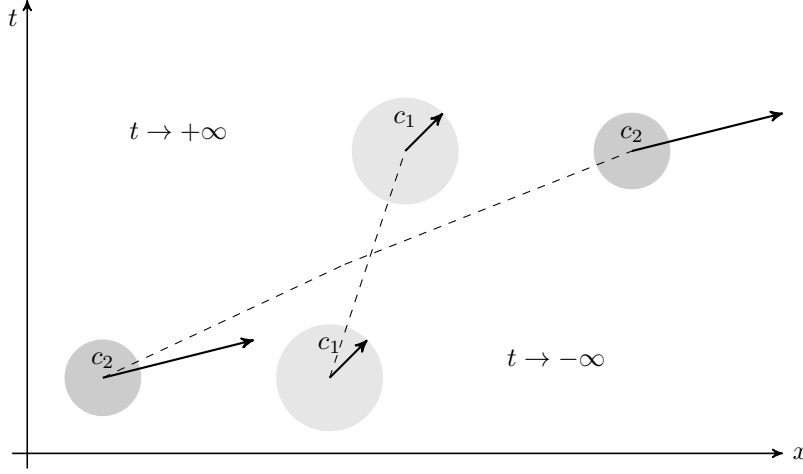


FIGURE 2. A 2-soliton for KdV, with $c_1 < c_2$. Note that the solution has the same scalings as $t \rightarrow \pm\infty$.

Remark 2.1. Contrary to the previous lecture, the energy space $H^1(\mathbb{R})$ is not well suited for the stability of more than one soliton, as we will see later. In that sense, any stability result in the energy space for the sum of several solitons is a difficult problem to solve.

Remark 2.2. If $p = 2, 3$, it is possible to find *explicit* solutions $U_N(t)$ of (1.1) such that

$$\lim_{t \rightarrow \pm\infty} \|U_N(t) - \sum_{j=1}^N Q_{c_j}(\cdot - c_j t - x_j^\pm)\|_{H^1(\mathbb{R})} = 0, \quad (2.5)$$

for $c_j > 0$ given and $x_j^\pm \in \mathbb{R}$ explicitly defined (actually, for $p = 2, 3$ the equation is *completely integrable*, see e.g. [31]). Such solutions are often referred as *multi-solitons*, or *N-soliton solutions*, and describe the interaction (collision) of N different solitons through the evolution in time. It is important to emphasize that, since (1.1) is a nonlinear equation, $U(t)$ is always different to the *sum* of N solitons, and (2.5) holds only at infinity in time. In particular, a good understanding of a property as (2.4) is a key step to understand the stability of $U_N(t)$ *globally* in time (see Fig. 2).

Remark 2.3. From numerical observations it is clear that a result like (2.4) cannot hold for the case $p = 4$ (the equation is in fact *nonintegrable*), unless the x_j^0 in (2.2) are well-ordered, in the sense that

$$x_1^0 < x_2^0 < \dots < x_N^0. \quad (2.6)$$

In fact, under this condition, *multi-collisions* are avoided (see Fig. 3).

The first stability result in the case of multi-solitons was proved by Maddocks and Sachs [34], based on an early work by Lax [32]. More precisely, they showed that for $p = 2$ (KdV), the multi-soliton $U_N(t)$ in (2.5) is stable under small perturbations in $H^N(\mathbb{R})$ (see [34] for a precise description of the result). Next, Martel, Merle and Tsai showed that the sum of N -solitons as in (2.3) is H^1 stable. More precisely,

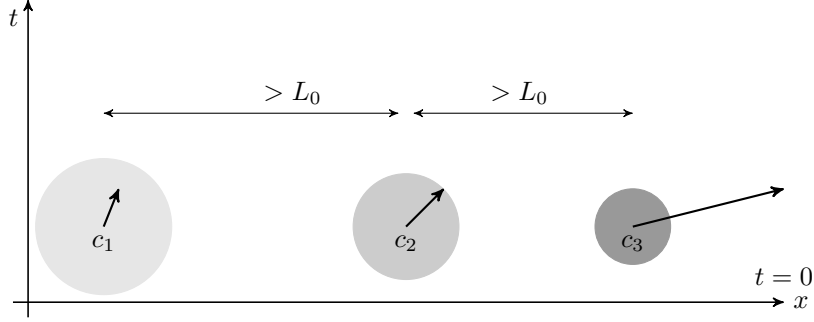


FIGURE 3. Well-prepared initial data, where $c_1 < c_2 < c_3$.

Theorem 2.1 ([39]). *Assume that $p = 2, 3$ or 4 , and that (2.1) and (2.6) holds, such that additionally*

$$\inf_j (x_{j+1}^0 - x_j^0) \geq L_0 > 0, \quad (2.7)$$

for some $L_0 > 0$ large. Then there are $C_0 > 0$ and $\alpha_0, \gamma_0 > 0$ such that for all $\alpha \in (0, \alpha_0)$, the following holds. Assume that $u_0 \in H^1(\mathbb{R})$ obeys the estimate

$$\|u_0 - \sum_{j=1}^N Q_{c_j}(\cdot - x_j^0)\|_{H^1(\mathbb{R})} < \alpha, \quad (2.8)$$

then the solution $u(t)$ of (1.2) with initial data u_0 satisfies

$$\sup_{t>0} \|u(t) - \sum_{j=1}^N Q_{c_j}(\cdot - \rho_j(t))\|_{H^1(\mathbb{R})} < C_0(\alpha + e^{-\gamma_0 L_0}). \quad (2.9)$$

Some remarks are in order.

Remark 2.4. In the case $p = 4$ (quartic gKdV), estimate (2.9) holds only for $t > 0$. Moreover, condition (2.7) is essential.

Remark 2.5. Using (2.5), estimate (2.9) improves the Maddocks-Sachs result. Additionally, Theorem 2.1 does not use the *integrability* of the equation.

Remark 2.6. The previous theorem has been adapted to several dispersive models with soliton solutions, see e.g. Martel-Merle-Tsai [39] for the case of the NLS equation, Muñoz [51] for the case of the defocusing mKdV equation, among others.

Remark 2.7. Recently, Alejo, Muñoz and Vega [7] showed that (2.9) holds even for L^2 perturbations, in the case $p = 2$ (KdV). In fact, the proof generalizes the Martel-Vega's original result.

2.1. Proof of Theorem 2.1. For simplicity, we assume $N = 2$. We define

$$R(t, x) := Q_{c_1}(x - \rho_1(t)) + Q_{c_2}(x - \rho_2(t)), \quad 0 < c_1 < c_2,$$

for some $\rho_1(t), \rho_2(t)$ to be chosen later. As in the previous section, the idea is to understand the behavior of the energy of $u(t)$ in the vicinity of $R(t)$, since $u(0) \sim Q_{c_1}(x - x_1^0) + Q_{c_2}(x - x_2^0)$.

Let us assume that $u(t) = R(t) + z(t)$, for some $z(t)$ small. We have ($z = z(t)$)

$$\begin{aligned} E[u] &= E[R] + \int_{\mathbb{R}} R_x z_x - \int_{\mathbb{R}} R^p z \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} z_x^2 - \frac{p}{2} \int_{\mathbb{R}} R^{p-1} z^2 + O(\|z\|_{H^1(\mathbb{R})}^3). \end{aligned} \quad (2.10)$$

We will assume now that ρ_1 and ρ_2 are well-separated, at least for a minimum amount of time. Indeed, assume that there is $c_0 > 0$ such that

$$\rho_2(t) - \rho_1(t) \geq c_0(t + L_0). \quad (2.11)$$

Using a standard bootstrap argument (in time), this hypothesis will be improved afterwards.

Now we consider the first order term in (2.10). We have

$$\int_{\mathbb{R}} R_x z_x - \int_{\mathbb{R}} R^p z = - \int_{\mathbb{R}} (R_{xx} + R^p) z.$$

Note that from the equation satisfied by each soliton Q_{c_j} , we have

$$R_{xx} + R^p = -c_1 Q_{c_1} - c_2 Q_{c_2} + [(Q_{c_1} + Q_{c_2})^p - Q_{c_1}^p - Q_{c_2}^p].$$

Thanks to condition (2.11) the term between brackets above satisfies the estimate

$$\|(Q_{c_1} + Q_{c_2})^p - Q_{c_1}^p - Q_{c_2}^p\|_{L^2(\mathbb{R})} \lesssim e^{-\gamma_0(L_0+t)},$$

for some $\gamma_0 > 0$. So we have

$$\int_{\mathbb{R}} R_x z_x - \int_{\mathbb{R}} R^p z = - \int_{\mathbb{R}} (c_1 Q_{c_1} + c_2 Q_{c_2}) z + O(\|z\|_{L^2(\mathbb{R})} e^{-\gamma_0(L_0+t)}).$$

However, unlike the previous section, the mass cannot help us as usual, since

$$\begin{aligned} M[u](t) &= M[Q_{c_1} + Q_{c_2}] + \int_{\mathbb{R}} (Q_{c_1} + Q_{c_2}) z + \frac{1}{2} \int_{\mathbb{R}} z^2 \\ &= M[Q_{c_1}] + M[Q_{c_2}] + \int_{\mathbb{R}} (Q_{c_1} + Q_{c_2}) z + \frac{1}{2} \int_{\mathbb{R}} z^2 + O(e^{-\gamma_0(L_0+t)}), \end{aligned}$$

and $c_1 \neq c_2$. The only option that we have to continue is to impose that the directions associated to each soliton vanish for all time, i.e.

$$\int_{\mathbb{R}} Q_{c_1} z = \int_{\mathbb{R}} Q_{c_2} z = 0. \quad (2.12)$$

We would like to use the Implicit Function Theorem to ensure that these orthogonality conditions do hold. A simple computation shows that we cannot modulate in time the variables ρ_1 and ρ_2 to obtain (2.12) mainly because the direction Q_{c_j} is even and the direction associated to ρ_j is odd ($= Q'_{c_j}$).

The idea now is to make c_1 and c_2 time-dependent functions, i.e. we will modulate the scalings. It is not difficult to check that thanks to (1.21) we can choose $c_1(t)$ and $c_2(t)$ close to the original scalings that we denote know c_1^0 and c_2^0 , such that now

$$\int_{\mathbb{R}} Q_{c_1(t)}(x - \rho_1(t)) z(t, x) dx = \int_{\mathbb{R}} Q_{c_2(t)}(x - \rho_2(t)) z(t, x) dx = 0, \quad (2.13)$$

and shift parameters $\rho_1(t)$ and $\rho_2(t)$ satisfying

$$\int_{\mathbb{R}} Q'_{c_1(t)}(x - \rho_1(t)) z(t, x) dx = \int_{\mathbb{R}} Q'_{c_2(t)}(x - \rho_2(t)) z(t, x) dx = 0, \quad (2.14)$$

where $z(t)$ is given now by

$$z(t, x) := u(t, x) - \tilde{R}(t, x), \quad \tilde{R}(t, x) := Q_{c_1(t)}(x - \rho_1(t)) + Q_{c_2(t)}(x - \rho_2(t)).$$

Note that this choice ensures (2.11), at least for certain amount of time. Coming back to (2.10), we have

$$\begin{aligned} E[u_0] &= E[\tilde{R}](t) + \frac{1}{2} \int_{\mathbb{R}} z_x^2(t) - \frac{p}{2} \int_{\mathbb{R}} \tilde{R}^{p-1} z^2(t) \\ &\quad + O(\|z(t)\|_{H^1(\mathbb{R})}^3 + e^{-\gamma_0(t+L_0)}). \end{aligned}$$

Note that now $E[\tilde{R}](t)$ is a function depending on time, so in some sense our problem has become even more difficult to manage, since the $O(1)$ terms in z are now time dependent.

Let us describe more carefully the term $E[\tilde{R}](t)$. It is not difficult to show that

$$E[\tilde{R}](t) = E[Q_{c_1(t)}] + E[Q_{c_2(t)}] + O(e^{-\gamma_0(t+L_0)}),$$

after using (2.11) and the fact that $c_1(t)$ and $c_2(t)$ are sufficiently close to the original scalings. Moreover, we have

$$E[Q_{c_j(t)}] = c_j^\theta(t) E[Q], \quad \theta := \frac{2}{p-1} + \frac{1}{2}. \quad (2.15)$$

For the sake of completeness, we also have

$$M[Q_{c_j(t)}] = c_j^{\tilde{\theta}}(t) E[Q], \quad \tilde{\theta} := \frac{2}{p-1} - \frac{1}{2}. \quad (2.16)$$

Now the problem will be to control the variation of the quantities $c_j^\theta(t)$ for all time.

Note that

$$\begin{aligned} 0 &= E[u](t) - E[u_0] = E[\tilde{R}](t) - E[\tilde{R}](0) + \frac{1}{2} \int_{\mathbb{R}} z_x^2(t) - \frac{p}{2} \int_{\mathbb{R}} \tilde{R}^{p-1} z^2(t) \\ &\quad + O(\|z(0)\|_{H^1(\mathbb{R})}^2 + \|z(t)\|_{H^1(\mathbb{R})}^3 + e^{-\gamma_0 L_0}) \end{aligned} \quad (2.17)$$

Now, using (2.15)

$$E[\tilde{R}](t) - E[\tilde{R}](0) = \sum_{j=1,2} (c_j^\theta(t) - c_j^\theta(0)) E[Q] + O(e^{-\gamma_0 L_0}). \quad (2.18)$$

From this identity and (2.17) we have for $j = 1, 2$,

$$|\sum_j \Delta c_j(t)| := |\sum_j c_j(t) - c_j(0)| \lesssim \|z(t)\|_{H^1(\mathbb{R})}^2 + e^{-\gamma_0 L_0}, \quad (2.19)$$

so $\sum_j \Delta c_j(t)$ varies *quadratically* on $z(t)$, and in particular it is smaller than $\|z(t)\|_{H^1(\mathbb{R})}$ (see [39] for a proof of the fact that each $\Delta c_j(t)$ is quadratic on z).

Here comes one of the new ideas in the Martel-Merle-Tsai's paper. Now we estimate (2.19) using (2.16): We have

$$c_j^\theta(t) - c_j^\theta(0) = \frac{(p+3)}{(5-p)} c_j(0) (c_j^{\tilde{\theta}}(t) - c_j^{\tilde{\theta}}(0)) + O(|\Delta c_j(t)|^2), \quad (2.20)$$

(recall that $\theta = \frac{2}{p-1} + \frac{1}{2}$ and $\tilde{\theta} = \frac{2}{p-1} - \frac{1}{2}$). Now we need an estimate on $c_j^{\tilde{\theta}}(t) - c_j^{\tilde{\theta}}(0)$. The second idea in the Martel-Merle-Tsai's paper is to consider a monotonicity property associated to the mass around the sum of N solitons.

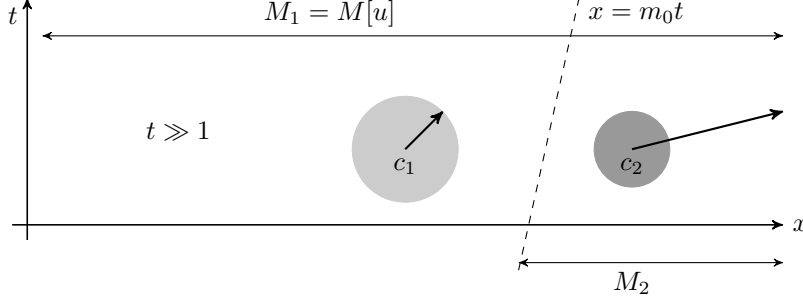


FIGURE 4. The portion of mass M_2 for two solitons. Scales are not proportional.

Define

$$m_0 := \frac{1}{2}(c_1^0 + c_2^0),$$

and for $A > 0$ large but fixed, we consider (see Fig. 4)

$$M_2(t) := \frac{1}{2} \int_{\mathbb{R}} u^2(t, x) \varphi\left(\frac{x - m_0 t}{A}\right),$$

where φ is a smooth kink satisfying

$$\varphi' > 0, \quad \varphi \in [0, 1], \quad \lim_{-\infty} \varphi = 0, \quad \lim_{+\infty} \varphi = 1,$$

and

$$|\varphi'''| \leq C\varphi'.$$

We can take for instance $\varphi(s) = \frac{2}{\pi} \arctan e^s$, so that

$$\varphi'(s) = \frac{2e^s}{\pi(1 + e^{2s})} = \frac{1}{\pi \cosh s}, \quad \varphi''(s) = \frac{-\sinh s}{\pi \cosh^2 s},$$

and

$$\varphi'''(s) = \frac{\sinh^2 s - 1}{\pi \cosh^3 s} = \frac{1}{\pi \cosh s} \cdot \frac{\sinh^2 s - 1}{\cosh^2 s},$$

so that $|\varphi'''(s)| \leq 2\varphi'(s)$.

Denote

$$y := \frac{x - m_0 t}{A}.$$

Using the fact that $m_0 t$ represents the *middle point* between both solitons and therefore the *supports* of $Q_{c_1}(x - \rho_1(t))$ and $\varphi\left(\frac{x - m_0 t}{A}\right)$ are largely disjoint, we have

$$\begin{aligned} M_2(t) &= \frac{1}{2} \int_{\mathbb{R}} (\tilde{R} + z)^2(t, x) \varphi(y) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} (Q_{c_2}^2 + 2Q_{c_2}z + z^2)(t, x) \varphi(y) dx + O(e^{-\gamma_0(t+L_0)}) \\ &= \frac{1}{2} \int_{\mathbb{R}} (Q_{c_2}^2 + z^2)(t, x) \varphi(y) dx + O(e^{-\gamma_0(t+L_0)}) \\ &= c_2^{\tilde{\theta}}(t) M[Q] + \frac{1}{2} \int_{\mathbb{R}} z^2(t, x) \varphi(y) dx + O(e^{-\gamma_0(t+L_0)}), \end{aligned} \quad (2.21)$$

for some $\gamma_0 > 0$ depending on A . Note that we have used (2.12) to get rid of the linear term on z .

On the other hand, following the original Kato smoothing estimate we have

$$\begin{aligned}
M'_2(t) &= -\frac{m_0}{2A} \int_{\mathbb{R}} u^2 \varphi' + \int_{\mathbb{R}} u u_t \varphi \\
&= \int_{\mathbb{R}} (u\varphi)_x (u_{xx} + u^p) - \frac{m_0}{2A} \int_{\mathbb{R}} u^2 \varphi' \\
&= -\frac{3}{2A} \int_{\mathbb{R}} u_x^2 \varphi' + \frac{p}{(p+1)A} \int_{\mathbb{R}} u^{p+1} \varphi' + \frac{1}{2A^3} \int_{\mathbb{R}} u^2 \varphi''' - \frac{m_0}{2A} \int_{\mathbb{R}} u^2 \varphi'.
\end{aligned} \tag{2.22}$$

Now we estimate (2.22). Since $u(t) = \tilde{R}(t) + z(t)$ we have

$$\begin{aligned}
\left| \int_{\mathbb{R}} u^{p+1} \varphi' \right| &\lesssim \int_{\mathbb{R}} |\tilde{R}|^{p+1} \varphi' + \int_{\mathbb{R}} |z|^{p+1} \varphi' + e^{-\gamma_0(L_0+t)} \\
&\lesssim \|z(t)\|_{H^1(\mathbb{R})}^2 \int_{\mathbb{R}} z^2 \varphi' + e^{-\gamma_0(L_0+t)}.
\end{aligned}$$

Recall that $\varphi'(y)$ and \tilde{R} have almost disjoint supports and the L^∞ norm is controlled by the H^1 norm in one dimension. We also have

$$\frac{1}{A^3} \left| \int_{\mathbb{R}} u^2 \varphi''' \right| \leq \frac{2}{A^3} \int_{\mathbb{R}} u^2 \varphi' \leq \frac{m_0}{4A} \int_{\mathbb{R}} u^2 \varphi',$$

provided A is large enough. Now (2.22) can be bounded as follows:

$$M'_2(t) \leq -\frac{3}{2A} \int_{\mathbb{R}} u_x^2 \varphi' - \frac{m_0}{4A} \int_{\mathbb{R}} u^2 \varphi' + C \|z(t)\|_{H^1(\mathbb{R})}^2 \int_{\mathbb{R}} z^2 \varphi' + C e^{-\gamma_0(L_0+t)},$$

so we get

$$M'_2(t) \leq C e^{-\gamma_0(L_0+t)}.$$

Integrating in time from $t = 0$ and any other time $t > 0$, we obtain the monotonicity estimate

$$M_2(t) \leq M_2(0) + C e^{-\gamma_0 L_0}.$$

We use now (2.21) to get

$$(c_2^{\tilde{\theta}}(t) - c_2^{\tilde{\theta}}(0))M[Q] + \frac{1}{2} \int_{\mathbb{R}} z^2 \varphi(t) \leq C \|z(0)\|_{H^1(\mathbb{R})}^2 + C e^{-\gamma_0 L_0}.$$

Consequently,

$$\begin{aligned}
-(c_2(0) - c_1(0))(c_2^{\tilde{\theta}}(t) - c_2^{\tilde{\theta}}(0))M[Q] &\geq \frac{1}{2}(c_2(0) - c_1(0)) \int_{\mathbb{R}} z^2 \varphi(t) \\
&\quad - C \|z(0)\|_{H^1(\mathbb{R})}^2 - C e^{-\gamma_0 L_0}. \tag{2.23}
\end{aligned}$$

On the other hand, using the conservation of mass we obtain

$$\begin{aligned}
&(c_1^{\tilde{\theta}}(t) - c_1^{\tilde{\theta}}(0))M[Q] + (c_2^{\tilde{\theta}}(t) - c_2^{\tilde{\theta}}(0))M[Q] \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} z^2(t) - \frac{1}{2} \int_{\mathbb{R}} z^2(0) + O(e^{-\gamma_0 L_0}) = 0,
\end{aligned}$$

or

$$\begin{aligned} -c_1(0)(c_1^{\tilde{\theta}}(t) - c_1^{\tilde{\theta}}(0) + c_2^{\tilde{\theta}}(t) - c_2^{\tilde{\theta}}(0))M[Q] &\geq \\ &\geq \frac{1}{2}c_1(0) \int_{\mathbb{R}} z^2(t) - C\|z(0)\|_{H^1(\mathbb{R})}^2 - Ce^{-\gamma_0 L_0}. \end{aligned} \quad (2.24)$$

Note that

$$\begin{aligned} c_1(0)(c_1^{\tilde{\theta}}(t) - c_1^{\tilde{\theta}}(0)) + c_2(0)(c_2^{\tilde{\theta}}(t) - c_2^{\tilde{\theta}}(0)) &= (c_2(0) - c_1(0))(c_2^{\tilde{\theta}}(t) - c_2^{\tilde{\theta}}(0)) + \\ &\quad + c_1(0)(c_1^{\tilde{\theta}}(t) - c_1^{\tilde{\theta}}(0) + c_2^{\tilde{\theta}}(t) - c_2^{\tilde{\theta}}(0)). \end{aligned}$$

Now we compute the energy of Q . We have from (1.3),

$$-\int_{\mathbb{R}} Q'^2 = \int_{\mathbb{R}} QQ'' = \int_{\mathbb{R}} Q^2 - \int_{\mathbb{R}} Q^{p+1}.$$

Multiplying (1.3) by Q and integrating, we have

$$\int_{\mathbb{R}} Q'^2 = \int_{\mathbb{R}} Q^2 - \frac{2}{p+1} \int_{\mathbb{R}} Q^{p+1}.$$

From these two identities, we have

$$\int_{\mathbb{R}} Q'^2 = \frac{2(p-1)}{p+3} M[Q], \quad \int_{\mathbb{R}} Q^{p+1} = 4 \frac{(p+1)}{p+3} M[Q].$$

Replacing in $E[Q]$, we obtain

$$E[Q] = \frac{1}{2} \int_{\mathbb{R}} Q'^2 - \frac{1}{p+1} \int_{\mathbb{R}} Q^{p+1} = \frac{p-5}{p+3} M[Q].$$

Note that the energy of a soliton is negative. We get the identity

$$\frac{(p+3)E[Q]}{(5-p)M[Q]} = -1.$$

Now (2.20) becomes

$$c_j^{\theta}(t) - c_j^{\theta}(0) = \frac{M[Q]}{|E[Q]|} c_j(0)(c_j^{\tilde{\theta}}(t) - c_j^{\tilde{\theta}}(0)) + O(\|z(t)\|_{H^1(\mathbb{R})}^4 + e^{-\gamma_0 L_0}).$$

Therefore, using (2.18), (2.20), (2.23) and (2.24)

$$\begin{aligned} E[\tilde{R}](t) - E[\tilde{R}](0) &= - \sum_{j=1,2} c_j(0)(c_j^{\tilde{\theta}}(t) - c_j^{\tilde{\theta}}(0))M[Q] \\ &\quad + O(e^{-\gamma_0 L_0} + \|z(t)\|_{H^1(\mathbb{R})}^4) \\ &\geq -(c_2(0) - c_1(0))(c_2^{\tilde{\theta}}(t) - c_2^{\tilde{\theta}}(0))M[Q] \\ &\quad - c_1(0)(c_1^{\tilde{\theta}}(t) - c_1^{\tilde{\theta}}(0) + c_2^{\tilde{\theta}}(t) - c_2^{\tilde{\theta}}(0))M[Q] \\ &\quad - Ce^{-\gamma_0 L_0} - C\|z(t)\|_{H^1(\mathbb{R})}^4 \\ &\geq \frac{1}{2}c_1(0) \int_{\mathbb{R}} z^2(t)(1 - \varphi(t)) + \frac{1}{2}c_2(0) \int_{\mathbb{R}} z^2(t)\varphi(t) \\ &\quad - C\|z(0)\|_{H^1(\mathbb{R})}^2 - Ce^{-\gamma_0 L_0}. \end{aligned}$$

Finally, from (2.17) and the previous estimate we find that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} z_x^2(t) + \frac{1}{2}c_1(0) \int_{\mathbb{R}} z^2(1 - \varphi)(t) + \frac{1}{2}c_2(0) \int_{\mathbb{R}} z^2\varphi(t) - \frac{p}{2} \int_{\mathbb{R}} \tilde{R}^{p-1} z^2(t) \\ \lesssim \|z(0)\|_{H^1(\mathbb{R})}^2 + \|z(t)\|_{H^1(\mathbb{R})}^3 + e^{-\gamma_0 L_0}. \end{aligned} \quad (2.25)$$

The left hand side of the last inequality can be decomposed in two parts corresponding to each soliton. Indeed, one has

$$\begin{aligned}\tilde{R}^{p-1}(t) &= \tilde{R}^{p-1}(1 - \varphi)(t) + \tilde{R}^{p-1}\varphi(t) \\ &= Q_{c_1}^{p-1}(1 - \varphi)(t) + Q_{c_2}^{p-1}\varphi(t) + O(e^{-\gamma_0 L_0}),\end{aligned}$$

and

$$z_x^2(t) = z_x^2(1 - \varphi)(t) + z_x^2\varphi(t),$$

so that if $w_1(t) := z(t)\sqrt{1 - \varphi(t)}$ and $w_2(t) := z(t)\sqrt{\varphi(t)}$, we get

$$\begin{aligned}z_x^2\varphi(t) &= \left(-\frac{w_2\varphi_x}{2\varphi^{3/2}} + \frac{(w_2)_x}{\sqrt{\varphi}} \right)^2 \varphi(t) \\ &= (w_2)_x^2(t) - \frac{w_2(w_2)_x\varphi_x}{\varphi} + \frac{w_2^2\varphi_x^2}{4\varphi^2}.\end{aligned}$$

Note that $|\varphi_x/\varphi| \leq C/A$, so for A large we have

$$\int_{\mathbb{R}} z_x^2\varphi(t) = \int_{\mathbb{R}} (w_2)_x^2(t) + O\left(\frac{1}{A}\|w_2(t)\|_{H^1(\mathbb{R})}^2\right).$$

Similarly,

$$\int_{\mathbb{R}} z_x^2(1 - \varphi)(t) = \int_{\mathbb{R}} (w_1)_x^2(t) + O\left(\frac{1}{A}\|w_1(t)\|_{H^1(\mathbb{R})}^2\right).$$

Replacing in (2.25), we obtain

$$\begin{aligned}&\frac{1}{2} \int_{\mathbb{R}} (w_1)_x^2(t) + \frac{1}{2} c_1(0) \int_{\mathbb{R}} w_1^2(t) - \frac{p}{2} \int_{\mathbb{R}} Q_{c_1}^{p-1} w_1^2(t) \\ &+ \frac{1}{2} \int_{\mathbb{R}} (w_2)_x^2(t) + \frac{1}{2} c_2(0) \int_{\mathbb{R}} w_2^2(t) - \frac{p}{2} \int_{\mathbb{R}} Q_{c_2}^{p-1} w_2^2(t) \\ &\lesssim \|z(0)\|_{H^1(\mathbb{R})}^2 + \|z(t)\|_{H^1(\mathbb{R})}^3 + e^{-\gamma_0 L_0}.\end{aligned}$$

Finally, we use the orthogonality conditions (2.13)-(2.14) as follows: note that

$$\begin{aligned}0 &= \int_{\mathbb{R}} Q_{c_2} z \\ &= \int_{\mathbb{R}} Q_{c_2} [z\sqrt{\varphi} + z(1 - \sqrt{\varphi})] \\ &= \int_{\mathbb{R}} Q_{c_2} w_2 + O(\|z(t)\|_{L^2(\mathbb{R})} e^{-\gamma_0 L_0}),\end{aligned}$$

with a constant depending on A (we can take L_0 larger if necessary). So w_2 is almost orthogonal to Q_{c_2} , and the error is small enough to be controlled using the formulation of coercivity described in (1.18). A similar argument allows to deal with every orthogonality condition. We finally get

$$\|w_1(t)\|_{H^1(\mathbb{R})}^2 + \|w_2(t)\|_{H^1(\mathbb{R})}^2 \lesssim \|z(0)\|_{H^1(\mathbb{R})}^2 + \|z(t)\|_{H^1(\mathbb{R})}^3 + e^{-\gamma_0 L_0}.$$

Finally, note that for A large,

$$\|w_1(t)\|_{H^1(\mathbb{R})}^2 + \|w_2(t)\|_{H^1(\mathbb{R})}^2 \sim \|z(t)\|_{H^1}^2.$$

Using this equivalence, we conclude.

3. THE COLLISION PROBLEM FOR NONINTEGRABLE gKdV EQUATIONS

In the previous Section (see (2.5)) we introduced the notion of N -soliton, or multi-soliton. The fact that the solution decomposes at infinity into the *same* original solitons (with different shifts only) is usually denoted as *elasticity*, that is, the collision among N solitons is elastic.

Recall that such a phenomenon is valid only for $p = 2$ and 3 , although the case $p = 3$ is a little bit more complicated to describe, as we will see in the next section. It is believed that no N -soliton solution exists for $p = 4$, mainly because the quartic gKdV equation is nonintegrable.

In 2005, Martel [35] proved the existence and uniqueness of *N -soliton-like* solutions for gKdV, $p = 2, 3$ and 4 . More precisely, there is a unique solution $U(t)$ of (1.1) such that

$$\lim_{t \rightarrow -\infty} \|U(t) - \sum_{j=1}^N Q_{c_j}(\cdot - c_j t - x_j^-)\|_{H^1(\mathbb{R})} = 0, \quad (3.1)$$

for (c_j) and x_j^- given. Clearly $U(t)$ coincides with the N -soliton solution $U_N(t)$ given in (2.5), for $p = 2$ and 3 . However, for $p = 4$ the behavior of $U(t)$ as $t \rightarrow +\infty$ is unknown, mainly because of the existence of a strong regime of interaction.

Martel's idea has shown to be a very robust technique to show existence of N -soliton-like solutions for a wide spectrum of dispersive models, mainly because the existence of such objects follows from compactness ideas, and it is not related to the stability of several solitons. In fact, it is possible to construct such solutions even if the corresponding single solitons are unstable! See e.g. [17] and [18].

Let us explain the ideas behind (3.1).

The proof is very similar to the stability proof described in the previous section. However, this time we do not need a monotonicity property since we are just constructing a particular family of solutions, and not describing the behavior of an open set of initial data.

From the fact that the interactions between different speed solitons decrease exponentially in time (due to the exponential decay of the soliton solution), Martel proved the uniform bound in time

$$\|U(t) - \sum_{j=1}^N Q_{c_j}(\cdot - c_j t - x_j^-)\|_{H^1(\mathbb{R})} \leq C e^{\gamma_0 t}, \quad t < 0, \quad (3.2)$$

for some $\gamma_0 > 0$ depending on the *minimal scaling*. Of course the bound above is interesting when t is largely negative, otherwise it loses its effectiveness.

In order to prove (3.2), the idea is as follows: we consider a decreasing sequence of times T_n approaching $-\infty$, and we solve the Cauchy problem for (1.2) with a particular set of initial data:

$$u_n(T_n) = R(T_n),$$

where

$$R(t, x) := \sum_{j=1}^N Q_{c_j}(x - c_j t - x_j^-).$$

Now the idea is to prove uniform estimates on n , for all time relatively large. More precisely, one has

$$\|u_n(t) - R(t)\|_{H^1(\mathbb{R})} \leq Ce^{\gamma_0 t}, \quad (3.3)$$

with constants independent of n , and for all $t \leq -T_0 < 0$ fixed. Note that this estimate is indeed valid for a certain amount of time near each T_n . Using the same idea as in the stability proof, one can bootstrap this estimate (without needing the monotonicity property at full) to conclude that (3.3) holds uniformly in time and the bounds are uniform on n . Finally, passing to the limit $n \rightarrow +\infty$ and using the continuity of the gKdV flow we get the desired estimate.

Once (3.1) is proved, the idea is to study the *collision* problem. Previous results in that direction can be found in Mizumachi [46], and some numerical simulations suggest that for $p = 4$, no elastic collision happens. In 2007, Martel and Merle considered a particular (simpler, but by no means less difficult) problem. They assumed $p = 4$, $N = 2$ (i.e. only two solitons), and additionally one soliton has to be smaller compared with the other one. Under this situation, they showed for the first time that the collision is indeed *inelastic*. They also consider [42] the case of general nonlinearities $f(u)$, proving stability of the two-soliton structure, but leaving open the question of inelasticity. Finally, we showed in [50] that no matter what the nonlinearity is (except for the integrable cases), the collision between two small solitons, one being even smaller compared with the other, is always inelastic.

Theorem 3.1 ([40, 42, 50]). *Assume $p = 4$ (quartic gKdV), $c_2 = 1$ and $c_1 = c \ll 1$. Consider the unique solution $U(t)$ satisfying*

$$\lim_{t \rightarrow -\infty} \|U(t) - Q(\cdot - t) - Q_c(\cdot - ct)\|_{H^1(\mathbb{R})} = 0.$$

Then there are $C_0 > 0$, $T_c \gg 1$, $c_1^+ > 1$ and $c_2^+ \in (0, c)$ (with $c_1^+ \sim 1$ and $c_2^+ \sim c$ in terms of powers of c smaller than $c^{11/12}$), and such that

(1) *Stability.*

$$\sup_{t \geq T_c} \|U(t) - Q_{c_1^+}(\cdot - \rho_1(t)) - Q_{c_2^+}(\cdot - \rho_2(t))\|_{H^1(\mathbb{R})} \leq C_0 c^{11/12}, \quad (3.4)$$

for some $\rho_1(t)$ and $\rho_2(t)$ satisfying the standard estimates.

(2) *Nonexistence.*

$$\liminf_{t \rightarrow +\infty} \|U(t) - Q_{c_1^+}(\cdot - \rho_1(t)) - Q_{c_2^+}(\cdot - \rho_2(t))\|_{H^1(\mathbb{R})} \geq \frac{1}{C_0} c^{17/12}. \quad (3.5)$$

Moreover, a similar result holds for any nonlinearity $f(u)$ in (1.1), provided

- (1) *f is subcritical around $u = 0$,*
- (2) *$f(u) \neq u^2, u^3$ and $u^2 + mu^3$, $m \in \mathbb{R}$, and*
- (3) *f has solitons Q_{c_1}, Q_{c_2} with $0 < c_2 \ll c_1 \ll 1$.*

Some remarks are in order.

Remark 3.1. The scaling c_1^+ and c_2^+ are unique in the sense of the *asymptotic stability* result:

$$\lim_{t \rightarrow +\infty} \|U(t) - Q_{c_1^+}(\cdot - \rho_1(t)) - Q_{c_2^+}(\cdot - \rho_2(t))\|_{H^1(x > \frac{1}{2}ct)} = 0. \quad (3.6)$$

If this condition is lifted, then c_2^+ and c_1^+ in (3.4) are not necessarily unique. Note that (3.6) says that in the region $x > \frac{1}{2}ct$ we have nothing except two solitons, at infinity.

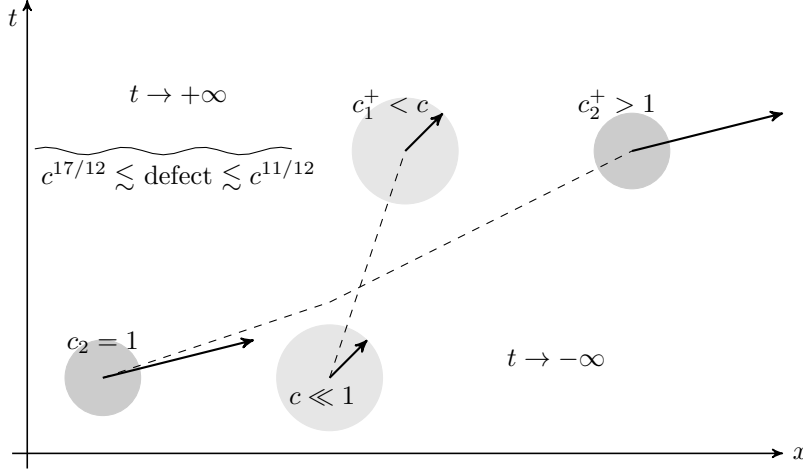


FIGURE 5. Theorem 3.1 for 2-soliton-like solution of the quartic gKdV equation, with $0 < c \ll 1$. Note that the solution has not the same scalings as $t \rightarrow \pm\infty$. The dashed lines are not necessarily straight lines. The defect is deeply related to the differences between final and original scalings.

Remark 3.2. In [41], Martel and Merle consider the case of collision between nearly equal solitons, proving similar results. In particular, the collision between two solitons is stable, but inelastic. For a similar account on this work, see e.g. [43].

Remark 3.3. The bounds $c^{11/12}$ and $c^{17/12}$ are inherent to the quartic case, in the general case where u^4 is replaced by $f(u)$ they may change. However, the difference between them (of the order $c^{1/2}$) seems to be always present. The understanding or improvement of this difference is a nice open problem.

The proof of Theorem 3.1 is involved and very long. The purpose of these notes is to give a suitable account of the main ideas of the proof of (3.4) and (3.5).

3.1. Sketch of proof of Theorem 3.1. Recall that from (3.2) we have for all $t < 0$,

$$\|U(t) - Q(\cdot - t) - Q_c(\cdot - ct)\|_{H^1(\mathbb{R})} \leq C_0 e^{\gamma_0 \sqrt{c} t}. \quad (3.7)$$

The constant \sqrt{c} appears in the exponential term after a careful analysis of the interaction between two solitons of size 1 and c in (3.2). From (3.7) it is clear that we will have a good control on the solution $U(t)$ provided $t \ll c^{-1/2}$. For this reason we define

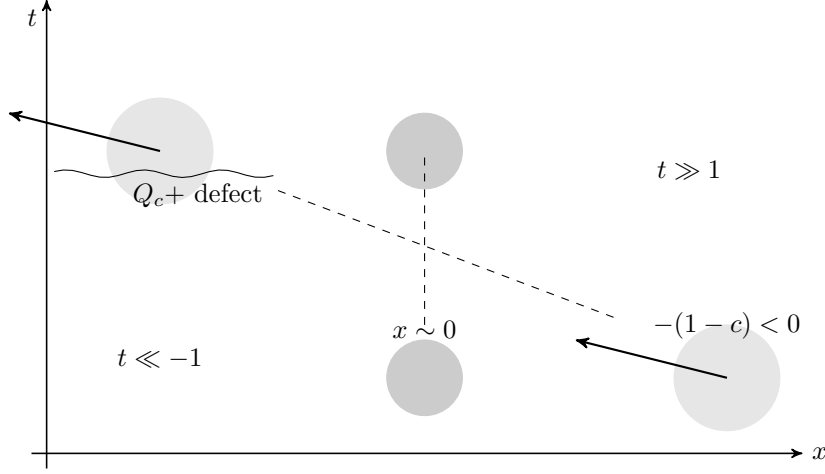
$$T_c := c^{-1/2 - \delta_0},$$

for some $\delta_0 > 0$ small but fixed. Note that we have

$$\begin{aligned} \|U(-T_c) - Q(\cdot + T_c) - Q_c(\cdot + cT_c)\|_{H^1(\mathbb{R})} &\leq C_0 e^{-\gamma_0 \sqrt{c} T_c} \\ &= C_0 e^{-\gamma_0 c^{-\delta_0}} \ll c^{1000}, \end{aligned} \quad (3.8)$$

for $c \ll 1$ small. In other words, the solution is almost the sum of two solitons at time $t = -T_c$. We denote by

$$[-T_c, T_c]$$


 FIGURE 6. Theorem 3.1 for the function $v(t, x)$

the interval of interaction in time (note that we do not know if the interaction lasts such an amount of time). We hope to have that at $t = T_c$,

$$\|U(T_c) - Q(\cdot - a_+) - Q_c(\cdot - b_+)\|_{H^1(\mathbb{R})} \lesssim c^{11/12}, \quad (3.9)$$

for some $a_+, b_+ \in \mathbb{R}$ satisfying $a_+ - b_+ \gg \frac{1}{2}T_c$. Note that estimate (3.9) makes sense since the soliton satisfies the estimate

$$\|Q_c\|_{H^1(\mathbb{R})} \sim c^{1/12},$$

namely, it is very much larger than the bound $c^{11/12}$. Assuming (3.9) and invoking Theorem 2.1 with c small, we will have (3.4).

Let us see how the proof of (3.9) works.

First of all, we will place ourselves at the origin in space. Let us define

$$v(t, x) := u(t, x + t).$$

It is not difficult to check that v satisfies the equation

$$S[v] := v_t + (v_{xx} - v + v^4)_x = 0. \quad (3.10)$$

The collision problem for v is composed by two solitons, $Q(x)$ fixed at the origin and a small soliton with negative velocity $Q_c(x + (1 - c)t)$ coming from positive infinite in space (see Fig. 6). Note that $(1 - c) > 0$, so the collision is in fact happening *fast* in time, but *slowly* in space, since the essential support of Q_c is of order $c^{-1/2}$. This last fact also justifies the choice of T_c as the time of interaction.

Let us introduce some important notation. We denote

$$y_c := x + (1 - c)t, \quad y := x - \alpha(y_c), \quad (3.11)$$

where α is a function to be chosen later. Recall that

$$S[v] := v_t + (v_{xx} - v + v^4)_x = 0.$$

Now we announce some easy-to-verify but very useful facts. Below γ_0 is a fixed small, positive constant, independent of c . We have

- (1) For $p = 4$, $Q_c(s) = c^{1/3}Q(\sqrt{c}s)$, and $Q_c'' - cQ_c + Q_c^4 = 0$ is the equation of the soliton.
 (2) For any $k \geq 1$ integer, we have

$$\|Q_c^k\|_{H^1(\mathbb{R})} \sim c^{\frac{k}{3}-\frac{1}{4}}, \quad \|(Q_c^k)'\|_{H^1(\mathbb{R})} \sim c^{\frac{k}{3}+\frac{1}{4}}. \quad (3.12)$$

- (3) Assume now that $f(x) \in \mathcal{S}(\mathbb{R})$, then

$$\|f(x)Q_c^k(y_c)\|_{H^1(\mathbb{R})} \sim c^{\frac{k}{3}}e^{-\gamma_0\sqrt{c}|t|}, \quad (3.13)$$

and

$$\|f(x)(Q_c^k)'(y_c)\|_{H^1(\mathbb{R})} \sim c^{\frac{k}{3}+\frac{1}{2}}e^{-\gamma_0\sqrt{c}|t|}. \quad (3.14)$$

Now we propose a first ansatz, say v_0 , for an approximate solution. Assume

$$v_0(t, x) := Q(x) + Q_c(y_c).$$

Then, using (1.3),

$$\begin{aligned} S[v_0] &= ((Q + Q_c)^4 - Q^4 - Q_c^4)_x \\ &= (4Q^3)'Q_c + 4Q^3Q_c' + (6Q^2)'Q_c^2 + 6Q^2(Q_c^2)' + 4Q'Q_c^3 + 4Q(Q_c^3)'. \end{aligned}$$

Note that from (3.13) we have that the worst term is actually the first one, since

$$\|(4Q^3)'Q_c\|_{H^1(\mathbb{R})} \lesssim c^{1/3}e^{-\gamma_0\sqrt{c}|t|}. \quad (3.15)$$

In what follows, we will make use of the following *principle of stability*:

Claim 1. If $\|S[\tilde{v}](t)\|_{H^1(\mathbb{R})} \lesssim c^m$ on $[-T_c, T_c]$, then the actual solution $v(t)$ should satisfy

$$\|v(t) - \tilde{v}(t)\|_{H^1(\mathbb{R})} \lesssim c^m T_c,$$

modulo a modulation in time on $v(t)$, and for all $t \in [-T_c, T_c]$.

This estimate can be seen as the effect of the propagation of the error term $S[v](t)$ through the interval of time $[-T_c, T_c]$. Note that this principle is in concordance with the case of a standard soliton $Q(x)$ which is stable for all time, since one has $S[Q] \equiv 0$.

For a moment we will assume the validity of this *pseudo theorem*, that we will prove later. Using this principle and (3.15), we should have

$$\|v(t) - \tilde{v}(t)\|_{H^1(\mathbb{R})} \lesssim c^{-1/12-\delta_0},$$

which is extraordinary large.

Now the goal is to get rid of the term $(4Q^3)'Q_c$. Using the algebra related to this term we will assume that

$$v_1(t, x) := Q(x) + Q_c(y_c) + A(x)Q_c(y_c),$$

with A unknown. Replacing in $S[\cdot]$ we obtain now

$$\begin{aligned} S[v_1] &= (1-c)AQ_c' + [(AQ_c)_{xx} - AQ_c + 4Q^3AQ_c]_x \\ &\quad + [(Q + Q_c + AQ_c)^4 - Q^4 - Q_c^4 - 4Q^3AQ_c]_x. \end{aligned}$$

Simplifying we obtain

$$\begin{aligned} S[v_1] &= Q_c(A'' - A + 4Q^3A + 4Q^3)' \\ &\quad + Q_c'(3A'' + 4Q^3A - cA) + \text{smaller terms}. \end{aligned} \quad (3.16)$$

Note that $A'' - A + 4Q^3A + 4Q^3 = 0$ can be written as (cf. (1.13) and Lemma 1.4)

$$(\mathcal{L}A)' = (4Q^3)', \quad \mathcal{L} := \mathcal{L}_{c=1},$$

so that we have a solution $A \in L^2(\mathbb{R})$ provided

$$\int_{\mathbb{R}} 4Q^3Q' = 0,$$

which is indeed the case. Moreover, it is clear that A is in the Schwartz class, and it has exponential decay. More precisely, A is explicitly given by the quantity

$$A = \frac{1}{3}Q' \int_0^x Q^2 - \frac{2}{3}Q^3. \quad (3.17)$$

In other words (see (3.13)),

$$\|A(x)Q_c(y_c)\|_{H^1(\mathbb{R})} \lesssim c^{1/3}e^{-\gamma_0\sqrt{c}|t|},$$

which implies that at time $t = T_c$ this term almost disappears, in other words, it cannot represent a defect appeared after the interaction.

Now we consider an improvement of v_1 , such that the second term in (3.16) disappears. Following the same idea as before, we consider

$$v_2(t, x) = Q(x) + Q_c(y_c) + A(x)Q_c(y_c) + B(x)Q'_c(y_c),$$

with B unknown. Replacing in the equation, we obtain the following equation for B :

$$(\mathcal{L}B)' = 3A'' + 4Q^3A + 4Q^3 \in \mathcal{S}(\mathbb{R}).$$

Note that not every term on the right above is the derivative of a localized function. Even worse, in order to have a solution for the previous linear equation we need

$$\int_{\mathbb{R}} (3A'' + 4Q^3A + 4Q^3)Q = 0,$$

since

$$\int_{\mathbb{R}} Q(\mathcal{L}B)' = - \int_{\mathbb{R}} Q' \mathcal{L}B = - \int_{\mathbb{R}} B \mathcal{L}Q' = 0.$$

However, we have

$$\int_{\mathbb{R}} (3A'' + 4Q^3A + 4Q^3)Q \neq 0.$$

Indeed, note that from (1.3) and (3.17),

$$\begin{aligned} \int_{\mathbb{R}} (3A'' + 4Q^3A + 4Q^3)Q &= \int_{\mathbb{R}} A(3Q'' + 4Q^4) + 4 \int_{\mathbb{R}} Q^4 \\ &= \int_{\mathbb{R}} A(3Q + Q^4) + 4 \int_{\mathbb{R}} Q^4 \\ &= 2 \int_{\mathbb{R}} Q^4 - \frac{2}{3} \int_{\mathbb{R}} Q^7 + \frac{1}{3} \int_{\mathbb{R}} (3Q + Q^4)Q' \int_0^x Q^2 \\ &= 2 \int_{\mathbb{R}} Q^4 - \frac{2}{3} \int_{\mathbb{R}} Q^7 - \frac{1}{3} \int_{\mathbb{R}} \left(\frac{3}{2}Q^2 + \frac{1}{5}Q^5\right)Q^2 \\ &= \frac{3}{2} \int_{\mathbb{R}} Q^4 - \frac{11}{15} \int_{\mathbb{R}} Q^7. \end{aligned}$$

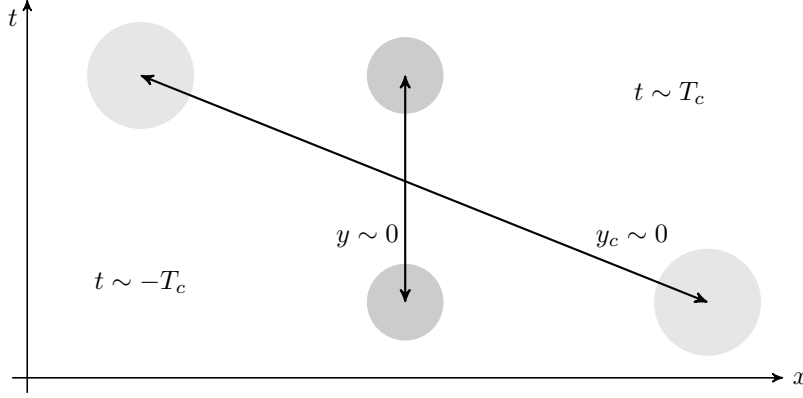


FIGURE 7. The two main variables for the collision problem: $y = x - \alpha(y_c)$ and $y_c = x + (1 - c)t$.

Finally, we have from (1.3),

$$\int_{\mathbb{R}} Q^4 = \int_{\mathbb{R}} Q, \quad \int_{\mathbb{R}} Q^7 = \frac{20}{11} \int_{\mathbb{R}} Q,$$

so that

$$\int_{\mathbb{R}} (3A'' + 4Q^3A + 4Q^3)Q = \frac{1}{6} \int_{\mathbb{R}} Q.$$

Key observation: We forgot the shift on $Q(x)$! Indeed, even in the integrable cases Q has a nontrivial shift. The idea is to introduce the variable y defined in (3.11), with a function α representing a shift (so it must vary from one nontrivial quantity to another one during a large period of time). In order to preserve the algebra already introduced, the key point is to take $\alpha(y_c)$ of the form

$$\alpha(y_c) = a_1 \int_0^{y_c} Q_c(s) ds.$$

With this choice any derivative of $Q(y)$ will give rise to a new term which is a suitable dilation of $Q_c(y_c)Q'(y)$, and it will preserve the multiplicative algebra. Similarly,

$$\partial_x Q(y) = Q'(y) - \alpha'(y_c)Q'(y) = Q'(y) - a_1 Q'(y)Q_c(y_c).$$

Note that the second term in the last equality is smaller compared with the first one, so the contribution of the shift will be at the level of the next linear problem. A similar reasoning is valid for $\partial_t Q(y)$. See Fig. 7 for the meaning of y and y_c .

We define

$$v_3(t, x) := Q(y) + Q_c(y_c) + A(y)Q_c(y_c) + B(y)Q'_c(y_c).$$

Now we look for B and $a_1 \in \mathbb{R}$. After a standard computation we obtain the perturbed linear system

$$(\mathcal{L}A)' + a_1(3Q - 2Q^4)' = (4Q^3)',$$

and

$$(\mathcal{L}B)' + a_1(3Q)'' - 3A'' - 4Q^3A = 4Q^3.$$

Recall that making $a_1 = 0$ we recover the original system, impossible to solve. However, now it is perfectly possible since

$$\int_{\mathbb{R}} Q'' Q = - \int_{\mathbb{R}} Q'^2 \neq 0.$$

However, the fact that the term $\mathcal{L}B$ has no localized right hand term implies that B must be chosen non localized. In fact, it is possible to solve the previous system as follows:

Lemma 3.2. *We have*

$$a_1 = -2 \frac{\int_{\mathbb{R}} Q}{\int_{\mathbb{R}} Q^2}, \quad B = -2\varphi, \quad \varphi := -\frac{Q'}{Q},$$

and

$$A = \frac{1}{3} Q' \int_0^x Q^2 - \frac{2}{3} Q^3 - a_1 \left(\frac{1}{3} Q - \frac{3}{2} y Q' \right).$$

Note that B is a *kink*, namely an odd function with negative derivative and limits at infinity. The uniqueness of A , B and a_1 has to be understood in the following sense: A is the unique even, localized solution, and B is odd.

Let us discuss the interpretation of B . Note that

$$B = -2\varphi(y) \rightarrow \mp 2 \quad \text{as } y \rightarrow \pm\infty,$$

so at time $t = -T_c$ we have that if $y_c \sim 0$ then $y \gg 1$, so

$$\begin{aligned} v_3(-T_c, x) &\sim Q(y) + Q_c(y_c) - 2Q'_c(y_c) \\ &\sim Q(y) + Q_c(y_c - 2). \end{aligned}$$

Similarly, at $t = T_c$ we have $y \ll -1$ if $y_c \sim 0$, and

$$\begin{aligned} v_3(T_c, x) &\sim Q(y) + Q_c(y_c) + 2Q'_c(y_c) \\ &\sim Q(y) + Q_c(y_c + 2). \end{aligned}$$

In other words, B represents the first order expansion of the shift appearing on the small soliton through the interaction.

The problem now is that we have not found an actual defect appearing from the interaction (something not related with shifts, scalings, etc.)

We perform a new ansatz. Now we take

$$\begin{aligned} v_4(t, x) &:= Q(y) + Q_c(y_c) + A_1(y)Q_c(y_c) + B_1(y)Q'_c(y_c) \\ &\quad + A_2(y)Q_c^2(y_c) + B_2(y)(Q_c^2)'(y_c), \end{aligned}$$

where $A_1 = A$ and $B_1 = B$ as before, A_2 and B_2 are unknown functions, and

$$y := x - \alpha(y_c), \quad \alpha(y_c) := a_1 \int_0^{y_c} Q_c(s) ds + a_2 \int_0^{y_c} Q_c^2(s) ds. \quad (3.18)$$

After replacing in the equation, we will obtain the following linear system for A_2 , B_2 and a_2 :

$$\begin{cases} (\mathcal{L}A_2)' + a_2(3Q - 2Q^4)' = F_2, \\ (\mathcal{L}B_2)' + 3a_2Q'' - 3A_2'' - 4Q^3A_2 = G_2. \end{cases} \quad (3.19)$$

Here

$$F_2 := (6Q^2(1 + A_1)^2)' - a_1(4Q^3 + 3A_1'' + 4Q^3A_1)' + 3a_1^2Q^{(3)};$$

and

$$G_2 := 6Q^2(1 + A_1)^2 + (6Q^2B_1(1 + A_1))' - \frac{1}{2}a_1(9A_1' + 3B_1'' + 4Q^3B_1) + \frac{3}{2}a_1^2Q^4.$$

Note that both elements, F_2 and G_2 are in the Schwartz class, but only F_2 has the structure of a derivative function of another Schwartz function. Note additionally that F_2 and G_2 depend on A_1 , B_1 and a_1 , which are already known.

In an independent work (see [44]), Martel and Merle discuss the solvability of this linear system. After several computations, they prove the following result.

Lemma 3.3. (3.19) has a solution (a_2, A_2, B_2) which satisfies $A_2 \in S(\mathbb{R})$, $B_2 = b\varphi + \hat{B}_2$, with $b < 0$ and $\hat{B}_2 \in S(\mathbb{R})$. In other words, $B_2 \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$.

Note that the new term $A_2(y)Q_c^2(y_c)$ is in some sense a *phantom term*, since after the interaction (namely, for $t \sim T_c$), it has almost no size. Indeed, we obtain

$$v_4(-T_c, x) \sim Q(y) + Q_c(y - 2) + b(Q_c^2)'(y_c),$$

and

$$v_4(T_c, x) \sim Q(y) + Q_c(y + 2) - b(Q_c^2)'(y_c).$$

Note that the term $(Q_c^2)'$ is supported near the small soliton.

Now we explain the main result in Martel-Merle's paper: the function B_2 represents a *defect* in the interaction. Let us explain why.

First of all, note that the term $(Q_c^2)'$ cannot be obtained from the natural shift and scaling variations on the big and small solitons. Indeed, if we have that the shift on the small soliton at time $t = T_c$ is Δ_c , then

$$Q_c(y_c + \Delta_c) = Q_c(y_c) + \Delta_c Q_c'(y_c) + \frac{1}{2}\Delta_c^2 Q_c''(y_c) + \dots$$

However, we cannot obtain the term $(Q_c^2)'$ using the derivatives of Q_c :

$$Q_c'' = cQ_c - Q_c^4,$$

$$Q_c^{(3)} = cQ_c' - (Q_c^4)',$$

and so on...In other words, the term $(Q_c^2)'$ does not appear. Similarly, if $\varepsilon = \varepsilon(y_c)$ is the variation of the scaling of the big soliton, we should have

$$Q_{1+\varepsilon}(y) = Q(y) + \Lambda Q(y)\varepsilon(y_c) + O_{H^1}(\varepsilon^2),$$

but every element above is very localized in the y variable (note that B_2 is just bounded). The same idea works for the case of the shift on the big soliton.

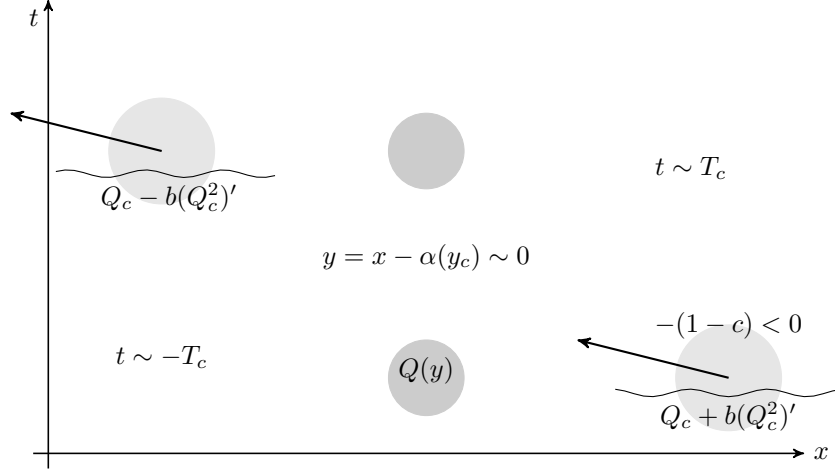
Finally, the best way to understand why B_2 represents a defect is by looking at the integrable cases. Note that the previous construction does not depend on the integrability of the equation, so it can be performed for the cases $p = 2$ and $p = 3$. For instance, if $p = 3$, we have

$$Q_c'' = cQ_c - Q_c^3,$$

$$Q_c^{(3)} = cQ_c' - (Q_c^3)',$$

so we expect that $b(p = 3) = 0$, which is indeed the case.

Lemma 3.4. If $p = 3$, one has $B_2 \in \mathcal{S}(\mathbb{R})$.

FIGURE 8. Identification of the defect for the symmetric function v_4 .

For the case of the KdV equation ($p = 2$), the interpretation is more subtle. We have that

$$Q_c^{(3)} = cQ_c' - (Q_c^2)',$$

so in fact $b \neq 0$, but it represents the third order variation of the shift on the small soliton, in the sense that

$$Q_c(y_c + \Delta_c) = Q_c(y_c) + \Delta_c Q_c'(y_c) + \frac{1}{2} \Delta_c^2 Q_c''(y_c) + \frac{1}{6} \Delta_c^3 Q_c^{(3)}(y_c) + \dots,$$

and $b = -\frac{1}{6} \Delta_c^3$ (note that this coincidence is the important point).

We conclude that every term of the form $b(Q_c^2)'(y_c)$ is *trivial* in the case of the integrable models. However, for the quartic case, it truly represents a *defect* (see Fig 8).

Let us compute the size of $(Q_c^2)'$: from (3.12) we have

$$\|(Q_c^2)'\|_{H^1(\mathbb{R})} \sim c^{2/3+1/4} = c^{11/12}.$$

This term will explain later the one in (3.4), because it is the first nontrivial element that appears from the interaction. However, in order to prove this fact, we need several additional improvements. First of all, note that after solving (3.19), the largest term in $S[v_4]$ is of the form

$$F(y)Q_c^3(y_c).$$

Even if we assume that $F \in \mathcal{S}(\mathbb{R})$ (the best case), we will have

$$\|F(y)Q_c^3(y_c)\|_{H^1(\mathbb{R})} \sim ce^{-\gamma_0\sqrt{c}|t|}, \quad \gamma_0 > 0.$$

Following the stability principle in Claim 1, we will have that

$$\|v(t) - v_4(t)\|_{H^1(\mathbb{R})} \lesssim c^{1/2}.$$

However, $c^{1/2} \gg c^{11/12}$, in other words, we will loose all control on the defect.

In [40] Martel and Merle continue solving even more complicated linear systems, however this time they use a general theory for solvability. An important problem

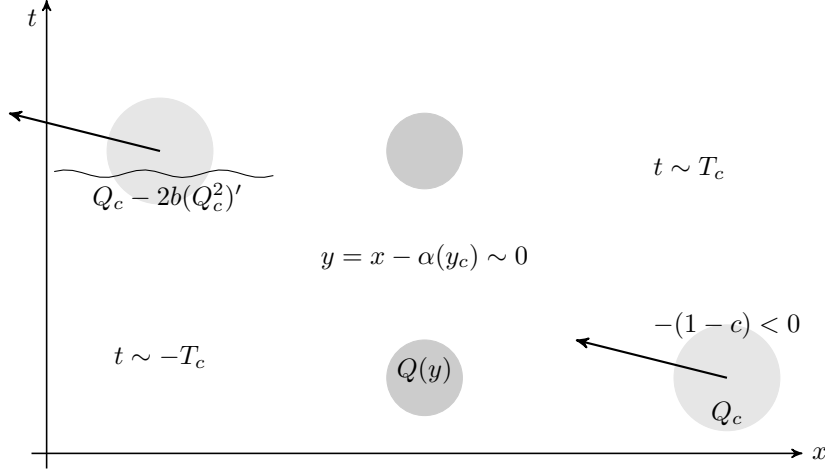


FIGURE 9. Identification of the true defect for the non symmetric function v_6 .

for that theory is the control of polynomially growing solutions on y (recall that B_2 is just bounded, and any equation involving the term $(\mathcal{L}B)'$ has no right hand side in a derivative form, so it may lead to large solutions). They use an approximate solution v_5 to solve up to order Q_c^5 , so that now

$$S[v_5] \sim F(y)Q_c^5, \quad F \in S(\mathbb{R}).$$

Moreover, v_5 is good enough, in the sense that its higher order terms do not grow too fast in the y variable. Note that

$$\|S[v_5](t)\|_{H^1(\mathbb{R})} \lesssim c^2 e^{-\gamma_0 \sqrt{c}|t|},$$

so we will have

$$\|v(t) - v_5(t)\|_{H^1(\mathbb{R})} \lesssim c^{3/2} \ll c^{11/12}.$$

Now the defect becomes evident for the dynamics.

Recall that up to this moment, we have just constructed an approximate solution which has a defect at both sides of time $t = -T_c$ and $t = T_c$. In that sense, this approximate solution is *symmetric*. We actually need a solution which is almost pure at $t = -T_c$ and not pure at $t = T_c$. In order to obtain such a solution, one has to modify v_5 as follows:

$$\begin{aligned} v_6(t, x) := & Q(y) + Q_c(y_c) + A_1(y)Q_c(y_c) + B_1(y)Q'_c(y_c) \\ & + A_2(y)Q_c^2(y_c) + B_2(y)(Q_c^2)'(y_c) - b(1 + \bar{v}(y))(Q_c^2)'(y_c) + \dots \end{aligned}$$

(see Fig. 9).

Here $\bar{v} \in \mathcal{S}$ (the Schwartz class) and it satisfies

$$B_2(y) - b(1 + \bar{v}(y)) \sim 0 \quad \text{for } t = -T_c,$$

and

$$B_2(y) - b(1 + \bar{v}(y)) \sim -2b \quad \text{for } t = T_c.$$

Of course we will loose some accuracy on the approximate solution. In that sense the function $\tilde{v}(y)$ allows to loose the minimum degree of accuracy. In [40] Martel and Merle proved that

$$\|S[v_6](t)\|_{H^1(\mathbb{R})} \sim c^{3/2} e^{-\gamma_0 \sqrt{c}|t|},$$

so now we have

$$\|v(t) - v_6(t)\|_{H^1(\mathbb{R})} \lesssim c, \quad (3.20)$$

which is still better than $c^{11/12}$. The defect is now recovered and (3.4) is proved.

Let us explain more in detail the stability principle announced in Claim 1. The idea is to prove the following

Proposition 3.5. *Assume (3.8) and \tilde{v} approximate solution of (3.10) such that*

$$\|\tilde{v}(-T_c) - Q(\cdot) - Q_c(\cdot - (1-c)T_c)\|_{H^1(\mathbb{R})} \lesssim c^{10}, \quad (3.21)$$

and for all time $t \in [-T_c, T_c]$, and $\theta > \frac{1}{3}$,

$$\|S[\tilde{v}](t)\|_{H^1(\mathbb{R})} \lesssim c^\theta T_c^{-1}.$$

Then there are $\rho(t) \in \mathbb{R}$ such that

$$\|V(t, \cdot - \rho(t)) - \tilde{v}(t)\|_{H^1(\mathbb{R})} \lesssim c^\theta, \quad (3.22)$$

where $V(t, x) := U(t, x + t)$. Additionally, $\rho'(t)$ is small.

(In order to prove (3.20) Martel and Merle take $\theta = 1$.) The proof of this result goes as follows. At least, for a certain amount of time $t > -T_c$, one has that (3.22) is satisfied, mainly because of (3.21), (3.8) and the continuity of the gKdV flow.

The idea is to extend this property up to time $t = T_c$. In order to prove this fact, we assume that no matter what is $\rho(t)$, the maximal time for which (3.22) is satisfied is just $T^* < T_c$. Under this assumption, we will perturb a little bit $V(t)$ by a particular choice of shift $\rho_0(t)$ for which

$$\int_{\mathbb{R}} Q'(y) z(t, x) dx = 0, \quad (3.23)$$

where

$$z(t, x) := V(t, x + \rho_0(t)) - \tilde{v}(t, x). \quad (3.24)$$

It is not difficult to see that (3.23) can be ensured via the Implicit Function Theorem. Moreover, one has

$$|\rho'_0(t)| \lesssim \|z(t)\|_{H^1(\mathbb{R})},$$

with a constant not depending on $z(t)$. Now we will bootstrap (3.22). Indeed, consider the Lyapunov functional

$$\mathcal{F}(t) := \frac{1}{2} \int_{\mathbb{R}} z_x^2 + \frac{1}{2} \int_{\mathbb{R}} (1 + \alpha'(y_c)) z^2 - \frac{1}{5} \int_{\mathbb{R}} [(\tilde{v} + z)^5 - \tilde{v}^5 - 5\tilde{v}^4 z].$$

Note that the term $\alpha'(y_c)$ is small compared with the constant 1, and it is needed since from (3.18) one has $y = x - \alpha(y_c)$, and

$$\alpha(y_c) \sim \int_{\mathbb{R}} Q_c \sim c^{-1/6},$$

which is a very large perturbation of the soliton center. It is not difficult to see that \mathcal{F} satisfies the lower bound, uniform on z ,

$$\mathcal{F}(t) \geq \gamma_0 \|z(t)\|_{H^1(\mathbb{R})}^2 - \frac{1}{\gamma_0} \left| \int_{\mathbb{R}} Q(y) z \right|^2, \quad (3.25)$$

mainly because $\tilde{v}(t) \sim Q(y) + \text{small terms}$. On the other hand, we compute the derivative of \mathcal{F} . First, we have that $z(t)$ satisfies the equation

$$z_t + (z_{xx} - z + (\tilde{v} + z)^4 - \tilde{v}^4)_x + S[\tilde{v}] - \rho'_0(\tilde{v} + z)_x = 0. \quad (3.26)$$

Now,

$$\begin{aligned} \mathcal{F}'(t) &= \int_{\mathbb{R}} z_t (-z_{xx} + (1 + \alpha'(y_c))z - (\tilde{v} + z)^4 + \tilde{v}^4) \\ &\quad + \frac{1}{2}(1 - c) \int_{\mathbb{R}} \alpha''(y_c) z^2 - \int_{\mathbb{R}} \tilde{v}_t [(\tilde{v} + z)^4 - \tilde{v}^4 - 4\tilde{v}^3 z]. \end{aligned}$$

We replace (3.26) to obtain

$$\begin{aligned} \mathcal{F}'(t) &= \int_{\mathbb{R}} (z_{xx} - z + (\tilde{v} + z)^4 - \tilde{v}^4)_x (z_{xx} - (1 + \alpha'(y_c))z + (\tilde{v} + z)^4 - \tilde{v}^4) \\ &\quad - \int_{\mathbb{R}} S[\tilde{v}] (-z_{xx} + (1 + \alpha'(y_c))z - (\tilde{v} + z)^4 + \tilde{v}^4) \\ &\quad - \rho'_0 \int_{\mathbb{R}} (\tilde{v} + z)_x (z_{xx} - (1 + \alpha'(y_c))z + (\tilde{v} + z)^4 - \tilde{v}^4) \\ &\quad + \frac{1}{2}(1 - c) \int_{\mathbb{R}} \alpha''(y_c) z^2 - \int_{\mathbb{R}} \tilde{v}_t [(\tilde{v} + z)^4 - \tilde{v}^4 - 4\tilde{v}^3 z]. \end{aligned} \quad (3.27)$$

The key estimate above is the one for (3.27). We have

$$|(3.27)| \lesssim \|S[\tilde{v}](t)\|_{H^1(\mathbb{R})} \|z(t)\|_{H^1(\mathbb{R})} \lesssim c^\theta T_c^{-1} \|z(t)\|_{H^1(\mathbb{R})}.$$

Note that this estimate is good enough since it depends only linearly on $\|z(t)\|_{H^1(\mathbb{R})}$ and not quadratically! For the other terms, the idea is to get estimates of the form

$$T_c^{-1-\delta_0} \|z(t)\|_{H^1(\mathbb{R})}^2,$$

for some $\delta_0 > 0$, or other better estimates, see [40] for a detailed proof (at this point the term $\alpha'(y_c)$ is needed). Under these circumstances, we will have

$$|\mathcal{F}'(t)| \lesssim c^\theta T_c^{-1} \|z(t)\|_{H^1(\mathbb{R})} + T_c^{-1-\delta_0} \|z(t)\|_{H^1(\mathbb{R})}^2,$$

which, after integration, leads to the bound

$$|\mathcal{F}(t)| \lesssim |\mathcal{F}(-T_c)| + c^\theta \sup_t \|z(t)\|_{H^1(\mathbb{R})} + o_{c \rightarrow 0}(1) \sup_t \|z(t)\|_{H^1(\mathbb{R})}^2.$$

Using (3.25), we will obtain

$$\sup_t \|z(t)\|_{H^1(\mathbb{R})}^2 \lesssim c^{2\theta} + \sup_t \left| \int_{\mathbb{R}} Q(y) z \right|^2,$$

with constants independent of $z(t)$. Finally, the linear term $\int_{\mathbb{R}} Qz$ above can be estimated using the conservation of mass, as in the first section of these notes, which leads to (3.22).

Finally, some words about (3.5). The proof uses an argument by contradiction. If (3.5) were not true at infinity in time, we can use an stability argument, as the one showed in Section 2, but this time backwards in time, for the sum of two

solitons of size 1 and c , plus a small error term. The idea is to prove that the size of such an error term is preserved up to the time $t = T_c$. However, note that the constant involved in the coercivity property of Theorem 2 depends on the size of the smaller of both solitons, which in this case is \sqrt{c} . This implies that by using the stability argument backwards near the small soliton we will loose $c^{1/2}$ of accuracy (recall that $c^{11/12+1/2} = c^{17/12}$), leading to a bound of the form (see (3.4))

$$\|Q(y) + Q_c(y_c) - v_6(T_c)\|_{H^1(\mathbb{R})} \leq \alpha c^{11/12},$$

for any small constant $\alpha > 0$. However, this bound contradicts the existence of a defect of size $\sim c^{11/12}$.

4. STABILITY OF PARTICULAR SOLITON STRUCTURES. THE CASE OF BREATHERS

In this last chapter we will discuss some very recent results by Miguel A. Alejo and myself. We will place ourselves in an integrable setting, in particular, we will consider the modified Korteweg-de Vries (mKdV) equation

$$u_t + (u_{xx} + u^3)_x = 0, \quad u(t, x) \in \mathbb{R}, \quad (t, x) \in \mathbb{R}^2. \quad (4.1)$$

Recall that in the case of real-valued initial data, the associated Cauchy problem for (4.1) is globally well-posed for initial data in $H^s(\mathbb{R})$, for any $s > \frac{1}{4}$, see Kenig-Ponce-Vega [27], and Colliander, Keel, Staffilani, Takaoka and Tao [16]. Additionally, the (real-valued) flow map is not uniformly continuous if $s < \frac{1}{4}$ [28]. In order to prove this last result, Kenig, Ponce and Vega considered a very particular class of solutions of (4.1) called *breathers*, discovered by Wadati in [58].

Definition 4.1 (See e.g. [58, 31]). *Let $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbb{R}$ be fixed parameters. The mKdV breather is a smooth solution of (4.1) given explicitly by the formula*

$$B := B(t, x; \alpha, \beta, x_1, x_2) := 2\sqrt{2}\partial_x \left[\arctan \left(\frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right], \quad (4.2)$$

where

$$y_1 := x + \delta t + x_1, \quad y_2 := x + \gamma t + x_2, \quad (4.3)$$

and

$$\delta := \alpha^2 - 3\beta^2, \quad \gamma := 3\alpha^2 - \beta^2. \quad (4.4)$$

(See Fig. 10.)

Breathers are *oscillatory bound states*. They are periodic in time (after a suitable space shift) and localized in space. The parameters α and β are scaling parameters, x_1, x_2 are shifts, and $-\gamma$ represents the *velocity* of a breather (see Fig. 11). For a detailed account of the physics of breathers see e.g. [31, 1, 9, 2, 4] and references therein.

Even if the equation is completely integrable in nature, with suitable results on the evolution of well-prepared initial data (see Kruskal et al. [21], Lax [32], Schuur [56], among others), no rigorous result on the stability of these solutions was given. Numerical computations (see Gorria-Alejo-Vega [3]) showed that breathers are *numerically* stable. With Alejo, we give a detailed description of the dynamics around a breather solution. First of all, we proved the following

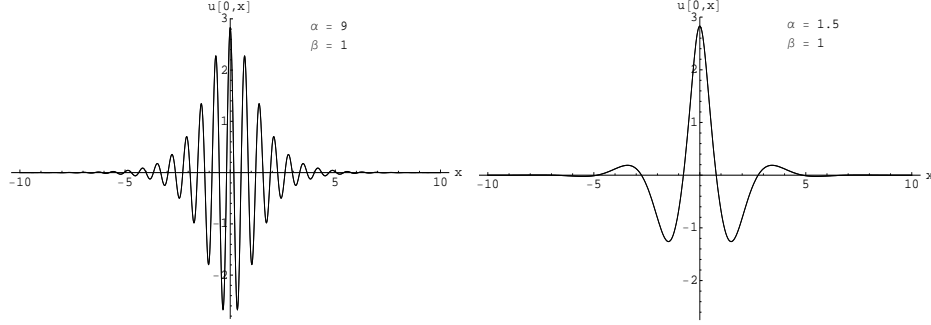


FIGURE 10. Left: mKdV breather (4.2) with $\alpha = 9, \beta = 1$ at $t = 0$. Right: mKdV breather (4.2) with $\alpha = 1.5, \beta = 1$ at $t = 0$.

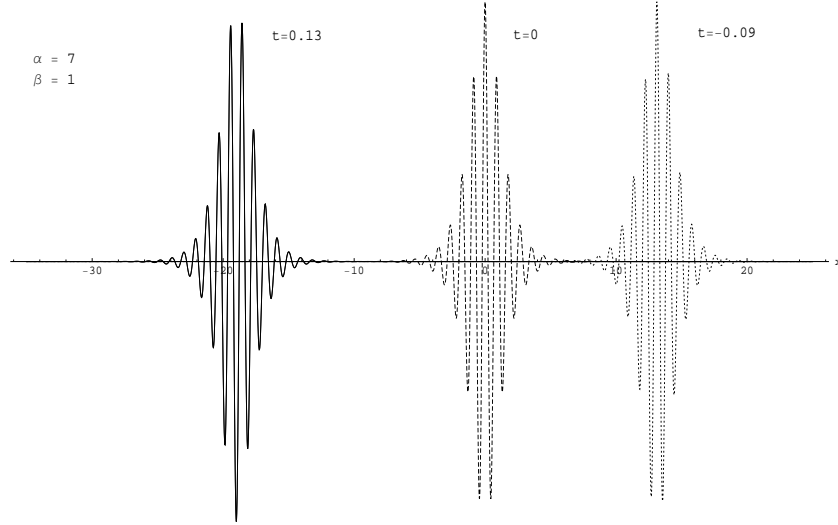


FIGURE 11. Evolution of the mKdV breather (4.2) with $\alpha = 7, \beta = 1$ at instants $t = -0.09, t = 0$, and $t = 0.13$. Note that with the selected values of α, β , the *velocity* is given by $\gamma = 3\alpha^2 - \beta^2 = 146 > 0$ and then the breather moves to the left. (Images taken from [4].)

Theorem 4.2 ([4]). *Breathers are H^2 -stable. More precisely, for any $\alpha, \beta > 0$, $x_1^0, x_2^0 \in \mathbb{R}$, there are $C_0 > 0$ and $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0)$ the following*

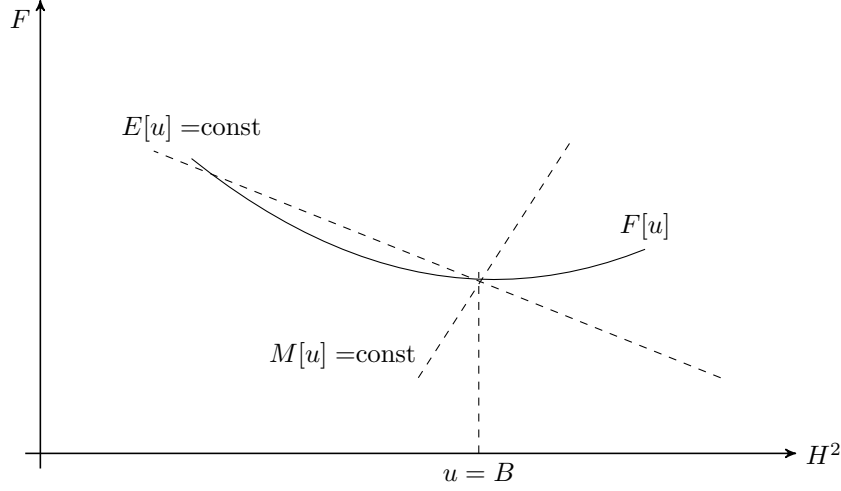


FIGURE 12. Schematic representation of Theorem 4.2. Each breather has a variational characterization; it is a (local) minimizer of the functional F defined in (4.5) under suitable constraints on its mass and energy.

holds. Assume that $u_0 \in H^2(\mathbb{R})$ satisfies

$$\|u_0 - B(0, \cdot; \alpha, \beta, x_1^0, x_2^0)\|_{H^2(\mathbb{R})} < \eta.$$

then there are $x_1(t), x_2(t) \in \mathbb{R}$ such that

$$\sup_{t \in \mathbb{R}} \|u(t) - B(t, \cdot; \alpha, \beta, x_1(t), x_2(t))\|_{H^2(\mathbb{R})} < C_0 \eta.$$

The proof of this result is in essence a variational one: we profit of the fact that breathers satisfy very special elliptic equations. For similar results in the case of KdV soliton solutions, see e.g. Lax [32] and Maddocks-Sachs [34].

4.1. Sketch of proof of Theorem 4.2. Since mKdV is an integrable equation, it has infinitely many conserved quantities. For the proof of Theorem (4.2) we will need the additional H^2 -conserved quantity

$$F[u](t) := \frac{1}{2} \int_{\mathbb{R}} u_{xx}^2(t) - \frac{5}{2} \int_{\mathbb{R}} u^2 u_x^2(t) + \frac{1}{4} \int_{\mathbb{R}} u^6(t) = F[u](0), \quad (4.5)$$

in addition to the standard mass and energy introduced in (1.10) and (1.11) (take $p = 3$). Consider the Lyapunov functional

$$H[u](t) := F[u](t) + 2(\beta^2 - \alpha^2)E[u](t) + (\alpha^2 + \beta^2)^2 M[u](t).$$

Clearly H is conserved for initial data $u_0 \in H^2(\mathbb{R})$. Moreover, any perturbation of a breather solution of the form

$$u(t, x) = B(t, x; \alpha, \beta, x_1(t), x_2(t)) + z(t, x),$$

with z small and $x_1(t), x_2(t)$ to be chosen later, must satisfy the expansion

$$H[u](t) = H[B] + \int_{\mathbb{R}} G[B](t)z + \frac{1}{2} \int_{\mathbb{R}} z \mathcal{L} z + O(\|z(t)\|_{H^2(\mathbb{R})}^3), \quad (4.6)$$

where $G[B]$ is the nonlinear operator

$$G[B] := B_{4x} - 2(\beta^2 - \alpha^2)(B_{xx} + B^3) + (\alpha^2 + \beta^2)^2 B \\ + 5BB_x^2 + 5B^2B_{xx} + \frac{3}{2}B^5,$$

and \mathcal{L} denotes the self-adjoint operator with domain $H^4(\mathbb{R})$:

$$\mathcal{L}z := z_{4x} - 2(\beta^2 - \alpha^2)z_{xx} + (\alpha^2 + \beta^2)^2 z + 5B^2z_{xx} + 10BB_xz_x \\ + (5B_x^2 + 10BB_{xx} + \frac{15}{2}B^4 - 6(\beta^2 - \alpha^2)B^2)z.$$

One of the key points of the proof is the fact that no matter what are x_1 and x_2 , one has

$$G[B] \equiv 0. \quad (4.7)$$

In other words, each breather satisfy a suitable fourth order elliptic equation, and in consequence B is a critical point for H (see Fig. 12). In order to prove this fact, one has two options, either computing (4.7) completely by hand, or proving simpler identities, as is done in [7]. First of all, from the definition (4.2) and (4.1) we have

$$\tilde{B}_t + B_{xx} + B^3 = 0. \quad (4.8)$$

Multiplying this equation by B_x and integrating in space we get

$$B_x^2 + \frac{1}{2}B^4 + 2B\tilde{B}_t - 2\mathcal{M}_t = 0, \quad (4.9)$$

where

$$\mathcal{M} := \frac{1}{2} \int_{-\infty}^x B^2.$$

The third identity that we will need is the following second order nonlocal equation

$$B_{xt} + 2\mathcal{M}_t B = 2(\beta^2 - \alpha^2)\tilde{B}_t + (\alpha^2 + \beta^2)^2 B, \quad (4.10)$$

which is the actual equivalent of (1.4) for the case of breathers. This last identity can be proved by hand (see [4] for a proof). Now we prove (4.7). We have from (4.9) and (4.8)

$$G[B] = -(B_t + B^3)_{xx} + 2(\beta^2 - \alpha^2)\tilde{B}_t + (\alpha^2 + \beta^2)^2 B \\ + 5BB_x^2 + 5B^2B_{xx} + \frac{3}{2}B^5 \\ = -B_{tx} - BB_x^2 + 2B^2B_{xx} + 2(\beta^2 - \alpha^2)\tilde{B}_t + (\alpha^2 + \beta^2)^2 B + \frac{3}{2}B^5 \\ = -B_{tx} + B \left[\frac{1}{2}B^4 + 2B\tilde{B}_t - 2\mathcal{M}_t \right] - 2B^2(\tilde{B}_t + B^3) + \frac{3}{2}B^5 \\ + 2(\beta^2 - \alpha^2)\tilde{B}_t + (\alpha^2 + \beta^2)^2 B \\ = -[B_{tx} + 2\mathcal{M}_t B] + 2(\beta^2 - \alpha^2)\tilde{B}_t + (\alpha^2 + \beta^2)^2 B = 0.$$

In the last line we have used (4.10).

Remark 4.1. Some interesting open questions arise from (4.7). Is B the unique localized solution to (4.7)? If not, under which conditions we recover the uniqueness? Note that for any x_1, x_2 shifts, the breather *profile* $B(0, x; \alpha, \beta, x_1, x_2)$ is solution to (4.7). In other words, there is a sort of two dimensional set of invariances for (4.7).

From the previous identity it is not difficult to show that

$$\mathcal{L}\partial_{x_1}B = \mathcal{L}\partial_{x_2}B = 0.$$

The fact that these two directions span the entire L^2 kernel of \mathcal{L} is not difficult to check. Indeed, the equation $\mathcal{L}z = 0$ is a fourth order ODE whose solutions are spanned by four linearly independent solutions. Their asymptotic at positive infinity are given by the forms

$$e^{\pm\beta x} \sin(\alpha x), \quad e^{\pm\beta x} \cos(\alpha x),$$

so unless $\beta = 0$, which is impossible by hypothesis, we only have two linearly independent localized solutions, which coincide with $\partial_{x_1}B$ and $\partial_{x_2}B$ above.

On the other hand, using the Weyl's theorem we have that the continuum spectrum is given by the intervals $[(\alpha^2 + \beta^2)^2, \infty)$ if $\beta \geq \alpha$, and $[4\alpha^2\beta^2, \infty)$ if $\beta \leq \alpha$.

We finally consider the problem of counting the number of negative eigenvalues. This is not an easy task, mainly because we deal with a fourth order ODE. The idea is to use the work by L. Greenberg [23], who shows that the number of negative eigenvalues,

$$\# \text{ negative eigenvalues} = \sum_{x \in \mathbb{R}} \dim \ker W[\partial_{x_1}B, \partial_{x_2}B](t, x),$$

where $W[\partial_{x_1}B, \partial_{x_2}B]$ is the Wronskian matrix associated to $\partial_{x_1}B$ and $\partial_{x_2}B$. The best way for understanding this identity is by considering the same problem for the case of a soliton solution. We have in this case

$$\# \text{ negative eigenvalues} = \sum_{x \in \mathbb{R}} \dim \ker Q'(x),$$

where we understand Q' as a linear operator in one dimension. Since $Q'(x) = 0$ only for $x = 0$, we have that $\sum_{x \in \mathbb{R}} \dim \ker Q'(x)$ is finite and equals one (i.e., we have just one negative eigenvalue), as expected from Lemma 1.4.

After some tedious computations we will obtain

$$\det W[\partial_{x_1}B, \partial_{x_2}B](t, x) = \frac{16\alpha^3\beta^3(\alpha^2 + \beta^2)(\alpha \sinh(2\beta y_2) - \beta \sin(2\alpha y_1))}{(\alpha^2 + \beta^2 + \alpha^2 \cosh(2\beta y_2) - \beta^2 \cos(2\alpha y_1))^2},$$

so we have nontrivial kernel if and only if

$$\alpha \sinh(2\beta y_2) = \beta \sin(2\alpha y_1). \quad (4.11)$$

Given x_1, x_2, t, α and β fixed, there is only one point $x = x_0 \in \mathbb{R}$ (depending on the previous parameters) for which this last identity holds. Indeed, fix x_1, x_2, t, α and β . We will look for \tilde{y}_2 solution of

$$\sinh(\tilde{y}_2) = \frac{\beta}{\alpha} \sin\left(\frac{\alpha}{\beta}\tilde{y}_2 + \tilde{x}_{12}\right), \quad (4.12)$$

where

$$\tilde{x}_{12} := 2\alpha[(\delta - \gamma)t + x_1 - x_2], \quad \tilde{y}_2 := 2\beta y_2.$$

(Recall that $y_1 = x + \delta t + x_1$, $y_2 = x + \gamma t + x_2$.) If $|\tilde{y}_2|$ is large enough, say $> M$, there is no solution for this equation. If now $|\tilde{y}_2| \leq M$, note that the function

$$[-M, M] \ni \tilde{y}_2 \mapsto \sinh(\tilde{y}_2) - \frac{\beta}{\alpha} \sin\left(\frac{\alpha}{\beta}\tilde{y}_2 + \tilde{x}_{12}\right) \in \mathbb{R} \quad (4.13)$$

changes its sign on $[-M, M]$, M large, so it has a root. Moreover, if $\tilde{x}_{12} \neq 2k\pi$, $k \in \mathbb{Z}$, such a root is unique ((4.13) has positive derivative). Now, if $\tilde{x}_{12} = 2k\pi$ for some $k \in \mathbb{Z}$, we will have

$$\sinh(\tilde{y}_2) = \frac{\beta}{\alpha} \sin\left(\frac{\alpha}{\beta} \tilde{y}_2\right),$$

for which $\tilde{y}_2 = 0$ is a root. If there is another one, it must be unique, by the same reason as before. We conclude that there is a unique root \tilde{y}_2 of (4.12). In particular, there is unique x_0 satisfying (4.11).

At the point x_0 we will have

$$1 \leq \dim \ker W[\partial_{x_1} B, \partial_{x_2} B](t, x_0) \leq 2,$$

but it is easy to see that the dimension cannot be 2 since the Wronskian matrix at that point is never identically zero.

We conclude that \mathcal{L} has a unique negative eigenvalue. It is possible to prove that such an eigenvalue is always far from zero, uniformly in time. Even better, following the ideas for the proof of (1.18), we are able to prove that there is $c_0 > 0$ only depending on α and β such that for all $z \in H^2(\mathbb{R})$, if

$$\int_{\mathbb{R}} z \partial_{x_1} B = \int_{\mathbb{R}} z \partial_{x_2} B = 0,$$

then

$$\int_{\mathbb{R}} z \mathcal{L} z \geq c_0 \|z\|_{H^2(\mathbb{R})}^2 - \frac{1}{c_0} \left| \int_{\mathbb{R}} B z \right|^2. \quad (4.14)$$

Using the conservation of mass we can estimate the last term above:

$$\begin{aligned} \left| \int_{\mathbb{R}} B z(t) \right| &\leq \left| \int_{\mathbb{R}} B z(0) \right| + \sup_{t \geq 0} \|z(t)\|_{L^2(\mathbb{R})}^2 \\ &\leq \|z(0)\|_{L^2(\mathbb{R})} + \sup_{t \geq 0} \|z(t)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Replacing in (4.14), and using (4.6), we conclude.

Remark 4.2. The ideas behind Theorem 4.2 are very robust and allow to prove different stability (instability) results, provided the linear problem satisfies the desired spectral properties. For example, the sine-Gordon equation (SG)

$$u_{tt} - u_{xx} + \sin u = 0, \quad (u, u_t)(t, x) \in \mathbb{R}^2, \quad (4.15)$$

has a breather solution (B, B_t) (here B_t represents the time derivative of B), whose explicit definition is not necessary for these notes. After some work, we were able to show that (B, B_t) satisfy the elliptic system of equations

$$B_{txx} + \frac{1}{8} B_t^3 + \frac{3}{8} B_x^2 B_t - \frac{1}{4} B_t \cos B - a B_t - \frac{b}{2} B_x = 0, \quad (4.16)$$

and

$$\begin{aligned} B_{(4x)} + \frac{3}{8} B_x^2 B_{xx} + \frac{3}{4} B_t B_{tx} B_x + \frac{3}{8} B_t^2 B_{xx} + \frac{5}{8} B_x^2 \sin B - \frac{5}{4} B_{xx} \cos B \\ + \frac{1}{4} \sin B \cos B - \frac{1}{8} B_t^2 \sin B - a(B_{xx} - \sin B) - \frac{b}{2} B_{tx} = 0, \end{aligned} \quad (4.17)$$

for some well-defined constants $a, b \in \mathbb{R}$. Additionally, there is an associated Lyapunov functional that control the dynamics for all time. See [5] for more details on these ideas.

4.2. H^1 stability. It turns out that the previous result can be improved to the level of allowing H^1 perturbations. However, now the proof is not variational, since the H^1 -stability will be a consequence of a *dynamical rigidity* for small perturbations of breather solutions associated to the integrability of the equation.

Theorem 4.3. *Breathers are H^1 stable, i.e. stable in the energy space.*

Sketch of proof. In order to prove this result, we need several preliminary definitions. First of all, we introduce the complex-valued mKdV soliton. Consider parameters $\alpha, \beta > 0$, x_1 and $x_2 \in \mathbb{R}$. Let

$$\tilde{Q} := \tilde{Q}(x; \alpha, \beta, x_1, x_2) := 2\sqrt{2} \arctan(e^{\beta y_2 + i\alpha y_1}), \quad (4.18)$$

where y_1 and y_2 are (re)defined as

$$y_1 := x + x_1, \quad y_2 := x + x_2. \quad (4.19)$$

We denote the complex-valued soliton profile by

$$Q := \partial_x \tilde{Q} = \frac{2\sqrt{2}(\beta + i\alpha)e^{\beta y_2 + i\alpha y_1}}{1 + e^{2(\beta y_2 + i\alpha y_1)}}. \quad (4.20)$$

Finally we denote

$$\tilde{Q}_t := -(\beta + i\alpha)^2 Q. \quad (4.21)$$

We remark that \tilde{Q} and Q may blow-up in finite time. Indeed, assume that

$$\text{for } x_2 \text{ fixed and some } k \in \mathbb{Z}, \quad x_1 = x_2 + \frac{\pi}{\alpha} \left(k + \frac{1}{2}\right). \quad (4.22)$$

Then \tilde{Q} and Q cannot be defined at $x = -x_2$. However, note that if $x_1 = x_2 = 0$, we have $Q(\cdot; \alpha, \beta, 0, 0) \in H^1(\mathbb{R}; \mathbb{C})$.

Another almost direct conclusion obtained from the definition of Q is the following

Lemma 4.4. *Fix $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbb{R}$ such that (4.22) is not satisfied. Then we have*

$$Q_{xx} - (\beta + i\alpha)^2 Q + Q^3 = 0, \quad \text{for all } x \in \mathbb{R}, \quad (4.23)$$

and

$$Q_x^2 - (\beta + i\alpha)^2 Q^2 + \frac{1}{2} Q^4 = 0, \quad \text{for all } x \in \mathbb{R}. \quad (4.24)$$

Moreover, the previous identities can be extended to any $x_1, x_2 \in \mathbb{R}$ by continuity.

Assume that (4.22) does not hold. Consider the sin and cos functions applied to complex numbers. We have from (4.18) and (4.20),

$$\begin{aligned} \sin\left(\frac{\tilde{Q}}{\sqrt{2}}\right) &= \sin(2 \arctan e^{\beta y_2 + i\alpha y_1}) \\ &= 2e^{\beta y_2 + i\alpha y_1} \cos^2(\arctan e^{\beta y_2 + i\alpha y_1}) \\ &= \frac{2e^{\beta y_2 + i\alpha y_1}}{1 + e^{2(\beta y_2 + i\alpha y_1)}} = \frac{1}{\beta + i\alpha} \frac{Q}{\sqrt{2}}. \end{aligned}$$

Similarly, from this identity we have

$$Q_x - (\beta + i\alpha) \cos\left(\frac{\tilde{Q}}{\sqrt{2}}\right) Q = 0,$$

so that from (4.21) and (4.24),

$$\begin{aligned} \tilde{Q}_t + (\beta + i\alpha) \left[Q_x \cos \left(\frac{\tilde{Q}}{\sqrt{2}} \right) + \frac{Q^2}{\sqrt{2}} \sin \left(\frac{\tilde{Q}}{\sqrt{2}} \right) \right] \\ = -(\beta + i\alpha)^2 Q + Q_x^2 Q^{-1} + \frac{1}{2} Q^3 = 0. \end{aligned}$$

So far, we have proved the following result.

Lemma 4.5. *Let Q be a complex-valued soliton profile with scaling parameters $\alpha, \beta > 0$ and shifts $x_1, x_2 \in \mathbb{R}$, such that (4.22) is not satisfied. Then we have*

$$\frac{Q}{\sqrt{2}} - (\beta + i\alpha) \sin \left(\frac{\tilde{Q}}{\sqrt{2}} \right) \equiv 0, \quad (4.25)$$

and

$$\tilde{Q}_t + (\beta + i\alpha) \left[Q_x \cos \left(\frac{\tilde{Q}}{\sqrt{2}} \right) + \frac{Q^2}{\sqrt{2}} \sin \left(\frac{\tilde{Q}}{\sqrt{2}} \right) \right] \equiv 0, \quad (4.26)$$

where $\sin z$ and $\cos z$ are defined on the complex plane in the usual sense.

We introduce now the notion of breather profile. Given parameters $x_1, x_2 \in \mathbb{R}$ and $\alpha, \beta > 0$, we consider y_1 and y_2 defined in (4.19). Let \tilde{B} be the kink profile

$$\tilde{B} = \tilde{B}(x; \alpha, \beta, x_1, x_2) := 2\sqrt{2} \arctan \left(\frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right), \quad (4.27)$$

and with a slight abuse of notation, we redefine

$$B := \tilde{B}_x. \quad (4.28)$$

Now we introduce the directions associated to the shifts x_1 and x_2 . Given a breather profile of parameters α, β, x_1 and x_2 , we define

$$B_1 = B_1(x; \alpha, \beta, x_1, x_2) := \partial_{x_1} B,$$

$$B_2 = B_2(x; \alpha, \beta, x_1, x_2) := \partial_{x_2} B.$$

and for δ and γ defined in (4.4),

$$\tilde{B}_t := \delta B_1 + \gamma B_2. \quad (4.29)$$

We also have from (4.8),

$$\tilde{B}_t + B_{xx} + B^3 = 0. \quad (4.30)$$

We will prove now that there is a deep interplay between complex solitons and breather profiles. Indeed,

Lemma 4.6. *Let (B, Q) be a pair breather-soliton profiles with scaling parameters $\alpha, \beta > 0$ and shifts $x_1, x_2 \in \mathbb{R}$. Assume that (4.22) is not satisfied. Then we have*

$$\frac{(B - Q)}{\sqrt{2}} - (\beta - i\alpha) \sin \left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}} \right) \equiv 0, \quad (4.31)$$

and

$$\tilde{B}_t - \tilde{Q}_t + (\beta - i\alpha) \left[(B_x + Q_x) \cos \left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}} \right) + \frac{(B^2 + Q^2)}{\sqrt{2}} \sin \left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}} \right) \right] \equiv 0. \quad (4.32)$$

Proof. Let us assume (4.31) and prove (4.32). We have from (4.21) and (4.23)

$$\tilde{Q}_t = -(\beta + i\alpha)^2 Q = -(Q_{xx} + Q^3).$$

Using (4.31) we have

$$B_x - Q_x - (\beta - i\alpha)(B + Q) \cos\left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}}\right) = 0,$$

and

$$\begin{aligned} B_{xx} - Q_{xx} - (\beta - i\alpha)(B_x + Q_x) \cos\left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}}\right) \\ + (\beta - i\alpha) \frac{(B + Q)^2}{\sqrt{2}} \sin\left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}}\right) = 0, \end{aligned}$$

so that using once again (4.31) and (4.30)

$$\begin{aligned} \tilde{B}_t - \tilde{Q}_t + (\beta - i\alpha) \left[(B_x + Q_x) \cos\left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}}\right) + \frac{(B^2 + Q^2)}{\sqrt{2}} \sin\left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}}\right) \right] \\ = -(B_{xx} + B^3) + Q_{xx} + Q^3 + \left[B_{xx} - Q_{xx} + (\beta - i\alpha) \frac{(B + Q)^2}{\sqrt{2}} \sin\left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}}\right) \right] \\ + (\beta - i\alpha) \frac{(B^2 + Q^2)}{\sqrt{2}} \sin\left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}}\right) \\ = Q^3 - B^3 + \sqrt{2}(\beta - i\alpha)(B^2 + Q^2 + BQ) \sin\left(\frac{\tilde{B} + \tilde{Q}}{\sqrt{2}}\right) \\ = Q^3 - B^3 + (B^2 + Q^2 + BQ)(B - Q) = 0. \end{aligned}$$

The proof of (4.31) is a tedious but straightforward computation which deeply requires the nature of the breather and soliton profiles. For the proof of this result, see [6, Appendix A]. \square

The previous properties are consequence of a deeper result. In what follows, we fix a primitive \tilde{f} of f , i.e.,

$$\tilde{f}_x := f, \tag{4.33}$$

where f is assumed only in $L^2(\mathbb{R})$. Notice that even if $f = f(t, x)$ is a solution of mKdV, then a corresponding term $\tilde{f}(t, x)$ may be unbounded in space. We introduce the spatial Bäcklund transformation [31]

$$G(u_a, u_b, m) := \frac{(u_a - u_b)}{\sqrt{2}} - m \sin\left(\frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}}\right). \tag{4.34}$$

Then

$$G(Q, 0, \beta + i\alpha) = 0. \tag{4.35}$$

and

$$G(B, Q, \beta - i\alpha) = 0. \tag{4.36}$$

There is a second component for this transformation G , which involves a time derivative, but for the sake of simplicity we will not deal with it. One can prove (see [6]) that given m , u_a is solution of mKdV provided u_b is another solution. For similar examples of the use of Bäcklund transformations, see e.g. the works by Mizumachi and Pelinovski [47], and Hoffman and Wayne [25].

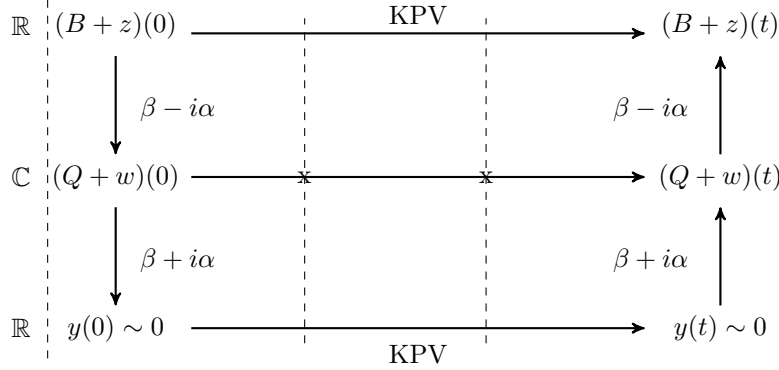


FIGURE 13. An oversimplified description of the proof of the H^1 stability of mKdV breathers, using and inverting the Bäcklund transformation twice with parameters $\beta - i\alpha$ and $\beta + i\alpha$. See [6] for a rigorous proof. KPV means Kenig, Ponce and Vega [27], the symbols \mathbb{R} and \mathbb{C} mean that the corresponding dynamics (the horizontal lines) are real-valued and complex-valued respectively, and the cross signs are the points where Q blows up (see (4.22)). Note that the fact that $y(0) \in H^1(\mathbb{R})$ is real-valued is a nontrivial property of the dynamics, consequence of the fact that $(B+z)(0)$ is real-valued, for $z(0) \in H^1(\mathbb{R})$ small enough.

From the previous paragraph, we see that the H^1 stability result will be a consequence of the Implicit Function Theorem applied to suitable H^1 neighborhoods of the points involved in (4.35) and (4.36), namely (B, Q) and $(Q, 0)$. Note that we are relating the breather to the zero solution via two Bäcklund transformations (see Fig 13).

The proof of this last result is involved, since (i) we have to prove two invertibility theorems, one near B and another near Q , (ii) we deal with initial data which is real valued, but the proof naturally introduces some complex-valued data, however, at the end we *have* to obtain real-valued data, (iii) this last property is a deep consequence of the particular form of the equation, and (iv) we must be careful with the times where Q blows up. We refer to the reader to [6] for a detailed proof.

Some final remarks. The previous results apply without important modifications to the case of the sine-Gordon (SG) equation (4.15) and its corresponding breather [31, p. 149]. See [11, 57, 19] and references therein for related results. Since the proofs are very similar, and in order to make this survey non redundant, we skip the details.

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